Minimal logics for reasoning with ambiguous expressions*

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Abstract

Alshawi and Crouch defined a simple multi-valued truth-conditional semantics for quasi logical form (QLF), a representation language for underspecified expressions. We incorporate this semantics within the setting of plain propositional logic, and investigate the underlying calculi.

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1 Introduction

The tasks of a semantic theory of underspecification are threefold. First, an adequate representation language which incorporates underspecification is required. Second, a resolution method has to be presented which maps every formula of the representation language to a set of formulas of a fully specified logical language. And third, such a theory needs a precise logical structure to enable reasoning with underspecified expressions: on the one hand models and a consequence relation for underspecified expressions have to be defined, and, on the other hand, a derivation system ought to be given

These three tasks do not make the natural language semanticist's life easy, on the contrary, each of these exercises is intrinsically hard. Early proposals of dealing with underspefications have merely focussed on the first two tasks: proper representation and resolution methods. In order to keep the complexity of the machinery under control, most of these systems focus on a single well-defined kind of underspecification. Most often this leads to easier resolutions, but increases the complexity of representation. General systems have often elegant representation languages, but it can be hard to define good algorithms for full resolution (if not impossible [22]).

The hope for proper logical semantics and calculi for such representation languages tends to fade away after confrontation with their immense complexity. In this paper we therefore take position at the other side of the gap that is to be bridged. We simply neglect representation and resolution of underspecified expressions, just naively presupposing that these tasks have been solved, and start construction works from the logical side.

The key purpose of the paper is to see what happens with ordinary propositional logic when an expression may have multiple readings. Since the logics only accept finite readings we speak of *ambiguous* instead of underspecified expressions [23].

It may be argued that a logic of ambiguity cannot stand on its own, and that reasoning about ambiguity simply requires representation and resolution [20]. We would agree on that point if we would restrict ourselves to a specific class of ambiguity. However, from our point of view a general system for reasoning with ambiguous expressions requires the minimalistic approach as explained above. The wide variety of representation languages and resolution methods shows that logical commitments towards representation and resolution seems to be impossible without loss of generality. In this perspective, specific logics incorporating a representation language and resolution methods come into focus after a minimal calculus for reasoning with ambiguous expresssions has been established.

1.1 Monotonic semantics

Monotonic semantics has been introduced by Alshawi and Crouch [3] as a truth-conditional model theory for so-called Quasi Logical Form (QLF), one of the well-known formalisms for representing underspecified natural language expressions [6]. As may be expected in view of its representation task, the language of QLF is very rich, and therefore, the achievement of monotonic semantics comes with proper truth-conditional decomposition of the wide variety of constructions in QLF.

Nevertheless, the basic idea of truth-value assignment for QLF is very simple, whenever the set of its full resolutions is given. An underspecified expression, i.e., a QLF with multiple LF readings, is true if *all* these specifications are true, and it is false if *all* these readings are false. This way of evaluating ambiguous expressions induces partiality. If the truth-values of the set of unambiguous readings are not all the same, then the original expression is left undefined. Furthermore, if a QLF has no readings, then this treatment induces overdefinedness, that is it is true and false at the same time.

The reason to call this monotonic semantic interpretation is that whenever an underspecified expression has been assigned a definite truth-value further specification will not change this evaluation.

In this paper we will focus on the logical foundation of this truth-conditional treatment. In order to be as clear as possible, we will restrict ourselves to a propositional setting.[1] The language consists of the normal language of propositional logic with an additional binary operator ? which simply refers to a two-fold ambiguity. The two arguments may themselves be ambiguous in turn. In this way we abstract away from the problem of resolving ambiguities, in order to concentrate on a propositional logical foundation for reasoning with ambiguities.

1.2 Maximal and relative ambiguity

Given the evaluation scheme for ambiguous expressions we may start to think about what a good entailment relation is. If we take *all* the readings into account we easily run into difficulties [5]. In [11, 7] an alternative so-called double barreled consequence relation has been defined, which will also be discussed in the first technical section of this paper.

Although this logic has some clear advantages, its 'maximal' treatment of ambiguity still yields some undesirable properties. In particular, the evaluation of expressions which have mutually contradictory reading causes great difficulties. Most of these irregularities can be removed when a relative notion of entailment is used over monotonic interpretations. Relativity here means that truth-value assignment to an expression also takes a (possibly partial) disambiguation of the expression into account. The idea of relative interpretation originates from a disambiguation dependent definition of reasoning in more recent descriptions of Reyle's Underspecified Discourse Representation Theory (UDRT) [25]. Reyle uses additional syntax, so-called co-indices, to describe this disambiguation dependence. Assumptions and conclusions are to be disambiguated in a synchronous fashion on the basis of principles of parallelism described by those co-indices. This specific technique will not be used in this paper, since it depends on the UDRT language.

We will show how a disambiguation dependent entailment over the truth-value assignment procedure of monotonic semantics removes most of the irregularities of the double barreled logic. Moreover, the resulting logic can be used as a starting point of defining stronger logics for reasoning with ambiguous expressions. Additional axioms or rules of inference correspond to constraints on the proces of disambiguation. We will present some of those logical correspondences at the end of the paper.

2 Preliminaries

The language The language $\mathcal{L}$ is the smallest set which contains an enumerable set of propositional variables $\mathbb{P}$ and for every $\varphi, \psi \in \mathcal{L}$ also $\neg\varphi$, $\varphi \wedge \psi \in \mathcal{L}$ and $?(\varphi, \psi) \in \mathcal{L}$. Other ordinary propositional connectives are defined in the standard manner, and we use $\perp$ for

[1] See [2] for first-order versions of the model-theory of monotonic semantics. In this paper there is no further analysis of logical validity over the models which are presented there. The paper focuses merely on the semantic treatment of meta-variables, dominantly present in QLF, in the setting of first-order logic.

$p \wedge \neg p$, $\top$ for $\neg(p \wedge \neg p)$ and $\star$ for $?(p, \neg p)$. We write $\mathcal{L}_0$ for the ?-free sublanguage of $\mathcal{L}$, i.e., the standard language of propositional logic.

Syntactic conventions $p, q, r, p', \ldots$ refer to propositional variables, while Greek lower case letters $\varphi, \psi, \varphi', \ldots$ refer to formulae from $\mathcal{L}$. Greek upper case letters $\Gamma, \Delta, \Gamma', \ldots$ refer to sets of $\mathcal{L}$-formulae, while Gothic upper case letters $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}, \mathfrak{X}', \ldots$ refer to sets of sets of $\mathcal{L}$-formulae.

We will use sets, and also sets of sets, of formulae as arguments of connectives. These expressions should be read distributively: $\neg\Gamma =_{df} \{\neg\varphi \mid \varphi \in \Gamma\}, \Gamma \wedge \Delta =_{df} \{\varphi \wedge \psi \mid \varphi \in \Gamma, \psi \in \Delta\}, \ldots, \neg\mathfrak{X} =_{df} \{\neg\Gamma \mid \Gamma \in \mathfrak{X}\}, \ldots$.

Inference In this paper we will present a variant of Gentzen-style sequent calculi[2] which are specially meant for the axiomatization of our 'quasi logics'. Instead of using sets[3] of formulae as arguments of the inference relation, we will present calculi in which these arguments are sets of finite sets of formulae, or shortly, *sfsf's* and *fsf's*, respectively. The underlying idea is that these finite sets represent a set of partial resolutions of some ambiguous expression. This reasoning format facilitates axiomatizations of the 'multiple-reading' logics of this paper. In ordinary Gentzen systems, the left-hand side of a sequent can be read as a big conjunction, while the right-hand side relates to a collective disjunction in the same way. Sequents of sfsf's make it possible to treat ambiguity in a similar manner, that is, every member of such an sfsf-argument can be seen as a compound ?-bound set. In this way, the reader should have the $? \wedge \ldots \wedge ? \vdash ? \vee \ldots \vee ?$ interpretation scheme for sequents in mind. This interpretation will be formalized in Definition 5 below, but first, let us give a precise definition of the Gentzen format derivations.

DEFINITION 1 *An sfsf-sequent is an expression of the form $\mathfrak{X} \vdash \mathfrak{Y}$ with $\mathfrak{X}$ and $\mathfrak{Y}$ being sfsf's. An sfsf-sequent rule is a rule of the form:*

$$\frac{\mathfrak{X}_1 \vdash \mathfrak{Y}_1 \quad \ldots \quad \mathfrak{X}_n \vdash \mathfrak{Y}_n}{\mathfrak{X} \vdash \mathfrak{Y}} \text{ with } n \in I\!N. \tag{1}$$

The sfsf-sequents $\mathfrak{X}_i \vdash \mathfrak{Y}_i$ are called the conditional *sfsf-sequents of the rule in (1), and $\mathfrak{X} \vdash \mathfrak{Y}$ is called its* conclusion *sfsf-sequent. If an* **s***-rule is unconditional — $n = 0$ in 1 — then we also speak of an* axiom rule. *In this case, we simply write $\mathfrak{X} \vdash \mathfrak{Y}$.*

An sfsf-sequent system **s** *is an enumerable set of sfsf-sequent rules. We say that $\mathfrak{Y}$ is* **s***-derivable from $\mathfrak{X}$, $\mathfrak{X} \vdash_{\mathbf{s}} \mathfrak{Y}$, if it can be established by a finite number of applications of the rules of* **s**. *If $\mathfrak{X} \vdash_{\mathbf{s}} \mathfrak{Y}$ and $\mathfrak{Y} \vdash_{\mathbf{s}} \mathfrak{X}$ then $\mathfrak{X}$ and $\mathfrak{Y}$ are called* **s**-interderivable *or* **s**-equivalent, *and we also write $\mathfrak{X} \equiv_{\mathbf{s}} \mathfrak{Y}$.*

Abbreviations $\Gamma \subseteq_f \Delta$ means that Γ is a finite subset of Δ. $\wp_f(\Delta)$ denotes the set of finite subsets of Δ.

We will limit the use of brackets in the presentation of sfsf-sequent systems. First, we do not use outer brackets for sfsf's, and empty sfsf's are just denoted by blanks, so $\vdash_{\mathbf{s}} \mathfrak{Y}$ means that $\mathfrak{Y}$ is **s**-derivable from the empty sfsf, and $\emptyset \vdash_{\mathbf{s}} \mathfrak{Y}$ means that $\mathfrak{Y}$ is derivable from

[2] See for example [26].

[3] We will not use sequences of formulae. We will not discuss any substructural logics here, and therefore, we prefer sets to avoid the superfluous use of standard structural rules.

the single sfsf which contains only the empty set. If $\vdash_{\mathbf{s}} \Gamma$ then Γ is an $\mathbf{s}$-theorem fsf, or an $\mathbf{s}$-tautological fsf. If $\vdash_{\mathbf{s}} \{\varphi\}$, then φ is an $\mathbf{s}$-tautology or $\mathbf{s}$-theorem.

In the same manner $\mathfrak{X} \vdash_{\mathbf{s}}$ and $\mathfrak{X} \vdash_{\mathbf{s}} \emptyset$ are to be distinguished. The former means that the empty sfsf can be derived from $\mathfrak{X}$, while the latter means that the singleton sfsf containing the empty set can be derived from $\mathfrak{X}$ in $\mathbf{s}$. The difference may be optically small, but the technical distinction is striking: $\mathfrak{X} \vdash_{\mathbf{s}}$ means that $\mathfrak{X}$ is in fact $\mathbf{s}$-inconsistent, while $\mathfrak{X} \vdash_{\mathbf{s}} \emptyset$ holds for all $\mathfrak{X}$ in most of the logics which we will present in this paper.[4]

Furthermore, we use $\mathfrak{X}, \mathfrak{Y}$ for $\mathfrak{X} \cup \mathfrak{Y}$, and $\mathfrak{X}, \Gamma$ instead of $\mathfrak{X} \cup \{\Gamma\}$. $\Gamma \cdot \varphi$ is used instead of $\Gamma \cup \{\varphi\}$.

Truth and falsity In the proceeding sections, we will define two forcing relations, $\models$ and $=\!\mid$, between each model class and the language $\mathcal{L}$ with usual infix notation. $M \models \varphi$ means that φ is *true* at M, while $M =\!\mid \varphi$ means that φ is *false* at M. The simplest class of models, which will be used throughout the paper, are ordinary *truth-value assignments*: $\mathsf{V} = I\!\!P \longrightarrow \{0,1\}$.

DEFINITION 2 *Truth and falsity for formulas with respect to a truth-value assignment V are defined in the following straightforward inductive manner:*

$$
\begin{array}{rcl@{\qquad}rcl}
V \models p & \Leftrightarrow & V(p) = 1 & V =\!\mid p & \Leftrightarrow & V(p) = 0 \\
V \models \neg\varphi & \Leftrightarrow & V =\!\mid \varphi & V =\!\mid \neg\varphi & \Leftrightarrow & V \models \varphi \\
V \models \varphi \wedge \psi & \Leftrightarrow & V \models \varphi \textit{ and } V \models \psi & V =\!\mid \varphi \wedge \psi & \Leftrightarrow & V =\!\mid \varphi \textit{ or } V =\!\mid \psi \\
V \models ?(\varphi,\psi) & \Leftrightarrow & V \models \varphi \textit{ and } V \models \psi & V =\!\mid ?(\varphi,\psi) & \Leftrightarrow & V =\!\mid \varphi \textit{ and } V =\!\mid \psi
\end{array}
$$

For an sfsf Γ we define:

$$V \models \Gamma \;\Leftrightarrow\; V \models \varphi \textit{ for all } \varphi \in \Gamma \qquad V =\!\mid \Gamma \;\Leftrightarrow\; V =\!\mid \varphi \textit{ for all } \varphi \in \Gamma$$

Note that $V \models \emptyset$ and $V =\!\mid \emptyset$. This coincides with the treatment of 'empty' expressions, expressions with no readings, as being overdefined in monotonic semantics [3].

It is easy to see how partiality is induced by the ambiguity operator: take $V(p) = 1$ and $V(q) = 0$ then $V \not\models ?(p,q)$ and $V \not=\!\mid ?(p,q)$. Note that there are many expressions which are always undefined. For example, $V \not\models \star$ and $V \not=\!\mid \star$ for all $V \in \mathsf{V}$.

Resolution sets In order to establish the relation between the ambiguity operator ? and the fsf's we define the resolution set $\mathsf{Res}(\varphi)$ of a formula φ as follows:

$$
\begin{array}{rcl}
\mathsf{Res}(p) & = & \{p\} \\
\mathsf{Res}(\neg\varphi) & = & \neg\,\mathsf{Res}(\varphi) \\
\mathsf{Res}(\varphi \wedge \psi) & = & \mathsf{Res}(\varphi) \wedge \mathsf{Res}(\psi) \\
\mathsf{Res}(?(\varphi,\psi)) & = & \mathsf{Res}(\varphi) \cup \mathsf{Res}(\psi)
\end{array}
$$

[4]All the systems in this paper are structural, and so, $\dfrac{\mathfrak{X} \vdash_{\mathbf{s}} \mathfrak{Y} \quad \mathfrak{Y} \subseteq \mathfrak{Y}'}{\mathfrak{X} \vdash_{\mathbf{s}} \mathfrak{Y}'}$ holds in general. Therefore $\mathfrak{X} \vdash_{\mathbf{s}}$ implies $\mathfrak{X} \vdash_{\mathbf{s}} \mathfrak{Y}$ for all $\mathfrak{Y}$. Furthermore, $\mathfrak{X} \vdash_{\mathbf{s}} \emptyset$ is a consequence of how we define truth for ambiguous expressions. The empty set represents the set of *no readings*, which is always true, and always false, since all the readings are true and false. The permanent truth of this finite set causes that it is a tautology in most of the logics of the paper. The only exception is the first logic to be presented in the next section. In this case, permanent falsity of $\emptyset$ causes it to be non-tautological.

Note that Res is a function from formulas to sets of formulas without ?-occurrences: $\mathsf{Res} : \mathcal{L} \longrightarrow \wp(\mathcal{L}_0)$. Furthermore, for fsf's Γ we also use:

$$\mathsf{Res}(\Gamma) \quad =_{df} \quad \bigcup_{\varphi \in \Gamma} \mathsf{Res}(\varphi).$$

A justification of this resolution function in terms of truth-value assignment is the following simple observation.

OBSERVATION 3 For every formula φ and fsf Γ, and every truth-value assignment Vs:

$$\begin{array}{ll} V \models \varphi \Leftrightarrow V \models \mathsf{Res}(\varphi) & V \models \Gamma \Leftrightarrow V \models \mathsf{Res}(\Gamma) \\ V \mathrel{=\!\!|} \varphi \Leftrightarrow V \mathrel{=\!\!|} \mathsf{Res}(\varphi) & V \mathrel{=\!\!|} \Gamma \Leftrightarrow V \mathrel{=\!\!|} \mathsf{Res}(\Gamma) \end{array}$$

Monotonicity From the truth-value assignment definition above it can easily be seen that all formulas are monotonic. All the connectives of the language preserve monotonicity.

OBSERVATION 4 For all $\varphi, \psi \in \mathcal{L}$ if $\mathsf{Res}(\varphi) \subseteq \mathsf{Res}(\psi)$ then for all $V \in \mathsf{V}$:

$$\begin{array}{lcl} V \models \psi & \Rightarrow & V \models \varphi \text{ and} \\ V \mathrel{=\!\!|} \psi & \Rightarrow & V \mathrel{=\!\!|} \varphi. \end{array}$$

For all fsf's $\Gamma \subseteq \Delta$ if $\mathsf{Res}(\Gamma) \subseteq \mathsf{Res}(\Delta)$ then for all $V \in \mathsf{V}$:

$$\begin{array}{lcl} V \models \Delta & \Rightarrow & V \models \Gamma \text{ and} \\ V \mathrel{=\!\!|} \Delta & \Rightarrow & V \mathrel{=\!\!|} \Gamma. \end{array}$$

Entailment For interpretation of sequents we will define an entailment relation in terms of a class of models M which reflect the semantic structures for the language $\mathcal{L}$. We will use different model classes for interpretation of M, but the notion of (single barreled) entailment will be fixed.

DEFINITION 5 *An sfsf $\mathfrak{X}$* entails *another sfsf $\mathfrak{Y}$ if for all models $M \in \mathsf{M}$ it holds that:*

$$\textit{if } M \models \Gamma \textit{ for all } \Gamma \in \mathfrak{X} \textit{ then } M \models \Delta \textit{ for certain } \Delta \in \mathfrak{Y}.$$

We write $\mathfrak{X} \models_{\mathsf{M}} \mathfrak{Y}$. $\mathfrak{X} \mathrel{=\!\!|}_{\mathsf{M}} \mathfrak{Y}$ means that rejection of all the premises enforces rejection of at least one of the conclusions, i.e., for all $M \in \mathsf{M}$:

$$\textit{if } M \mathrel{=\!\!|} \Gamma \textit{ for all } \Gamma \in \mathfrak{X} \textit{ then } M \mathrel{=\!\!|} \Delta \textit{ for certain } \Delta \in \mathfrak{Y}.$$

Functional completeness The truth-value assignment given in Definition 2 yields a three valued interpretation of the formulas of $\mathcal{L}$. A formula is true, false or neither true nor false (*undefined* truth-value). The possibility of being both true and false at the same time does not occur.

OBSERVATION 6 For all $V \in \mathsf{V}$ and $\varphi \in \mathcal{L}$ it is impossible that $V \models \varphi$ and $V \mathrel{=\!\!|} \varphi$. This property also holds for non-empty sets. Remember that only the empty set yields overdefinedness.

Proof. By a simple induction on the construction of formulae. ◁

This simple observation means that our connectives can be interpreted as three-valued Boolean functions, i.e., the set of functions

$$\{\text{undefined}, \text{true}, \text{false}\}^n \longrightarrow \{\text{undefined}, \text{true}, \text{false}\}.$$

Let us write the corresponding functions as $[\neg]$, $[\wedge]$ and $[?]$, then:

$$[\neg](t) = \begin{cases} \text{false} & \text{if } t = \text{true} \\ \text{true} & \text{if } t = \text{false} \\ \text{undefined} & \text{if } t = \text{undefined} \end{cases}$$

$$[\wedge](t_1, t_2) = \begin{cases} \text{true} & \text{if } t_1 = t_2 = \text{true} \\ \text{false} & \text{if } t_1 = \text{false or } t_2 = \text{false} \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$[?](t_1, t_2) = \begin{cases} \text{true} & \text{if } t_1 = t_2 = \text{true} \\ \text{false} & \text{if } t_1 = t_2 = \text{false} \\ \text{undefined} & \text{otherwise} \end{cases}$$

There is a striking relation between the monotonicity of our language and monotonicity of three-valued Boolean functions. Such a function f is called monotonic if whenever it has a definite value (true or false) for a given input, then this value does not change whenever we make the input more definite, i.e., replace undefined occurrences by definite values.[5]

THEOREM 7 (BLAMEY, 1984) *Every monotonic three-valued Boolean can be defined in terms of the functions* $[\neg]$, $[\wedge]$ *and* $[?]$.

Proof. This has been proved by Blamey in the context of partial (three valued) logic [4].[6] ◁

The conclusion which we may draw from this result of partial logic is that our language is not only monotonic, but that in the context of propositional logic, any other monotonic connective that we would like to add to the logic can be defined in terms of the three connectives that appear in $\mathcal{L}$. In other words, addition of a really new connective, undefinable by the connectives in $\mathcal{L}$, would necessarily lead to violation of the monotonicity criterion.

3 Maximal ambiguity

The first logic which we want to present considers just the simple model class V. This means that for every expression *all* its readings have to be taken into account when it comes to

[5] Technically speaking, these function are monotonic over the intuitive order on the three values: undefined < true and undefined < false, so, f is monotonic if $t_i \leq t'_i$ for all $i \in \{1, \ldots, n\}, t_i, t'_i \in \{\text{undefined}, \text{true}, \text{false}\}$ then $f(t_1, \ldots, t_n) \leq f(t'_1, \ldots, t'_n)$.

[6] Blamey uses the binary 'interjunction' which is defined in the same manner as our ambiguity operator ?. Blamey also uses the 0-ary connective $\top$, but we can define this by $\neg(p \wedge \neg p)$, a proposition which is always true. The clear difference is that in the case of partial logic, propositional variables can be undefined as well, while we enforce bivalence for variables.

truth-value assignment. In later sections we will discuss logics that leave the possibility of intermediate disambiguation open.

Application of the entailment relation of Definition 5 over the ordinary truth-value assignments gives immediately intuitively wrong results: $?(\varphi,\psi) \models_{\mathsf{V}} \varphi \wedge \psi$. It would be a much too strong conclusion that all readings follow from an ambiguous expression. In particular, any expression with mutually contradictory readings would lead to inconsistency. So, in general $?(\varphi,\psi) \not\models \varphi \wedge \psi$ is undesirable, when $\models$ is some notion of entailment over ambiguous expressions. On the other hand it seems desirable to obtain $?(\varphi,\psi) \models \varphi \vee \psi$. This means that we may be uncertain about which of the readings is the case, the conclusion that at least one of them should hold is safe. On the other hand, identification of ambiguous expressions with disjunction of the readings seems to be wrong, for example, it seems implausible to accept $\varphi \vee \psi \models ?(\varphi,\psi)$, and also its contraposition $\neg\,?(\varphi,\psi) \models \neg(\varphi \vee \psi)$ looks highly dubious [5]. Furthermore, $\varphi \wedge \psi \models ?(\varphi,\psi)$ is something one likes to have in a logic of ambiguous expressions. These simple observations lead to the following list of conjunctive and disjunctive desiderata:

$$
(2) \qquad
\begin{array}{rcl@{\qquad}rcl}
?(\varphi,\psi) & \models & \varphi \vee \psi & ?(\varphi,\psi) & \not\models & \varphi \wedge \psi \\
\varphi \wedge \psi & \models & ?(\varphi,\psi) & \varphi \vee \psi & \not\models & ?(\varphi,\psi) \\
\neg\,?(\varphi,\psi) & \models & \neg\varphi \vee \neg\psi & \neg\,?(\varphi,\psi) & \not\models & \neg\varphi \wedge \neg\psi \\
\neg\varphi \wedge \neg\psi & \models & \neg\,?(\varphi,\psi) & \neg\varphi \vee \neg\psi & \not\models & \neg\,?(\varphi,\psi).
\end{array}
$$

The single barreled entailment relation over V, $\models_{\mathsf{V}}$, establishes the entailments on the left hand side of the list. But it only satisfies the second and fourth requirement on the right hand side.

3.1 Double barreled entailment

In [7] a so-called *double barreled entailment relation* has been proposed as a good alternative to establish the desiderata mentioned above. This relation have been introduced in partial logic [4].

DEFINITION 8 *An sfsf* $\mathfrak{Y}$ *is said to be entailed by an sfsf* $\mathfrak{X}$ *in a* double barreled *fashion over a class of models* M *if* $\mathfrak{X} \models_{\mathsf{M}} \mathfrak{Y}$ *and* $\mathfrak{Y} =\!|_{\mathsf{M}} \mathfrak{X}$. *We abbreviate this relation by* $\mathfrak{X} \models^2_{\mathsf{M}} \mathfrak{Y}$.

Application of this definition to the model class V means that $\mathfrak{X} \models^2_{\mathsf{V}} \mathfrak{Y}$ iff $\mathfrak{X} \models_{\mathsf{V}} \mathfrak{Y}$ and $\neg\mathfrak{Y} \models_{\mathsf{V}} \neg\mathfrak{X}$. Note that this latter contrapositional variant does not follow from the former: $?(p,q) \models_{\mathsf{V}} p \wedge q$ but $\neg(p \wedge q) \not\models_{\mathsf{V}} \neg\,?(p,q)$.

This enforced contraposition restores symmetry in partial logics, which has great technical and meta-theoretical advantages [19]. Also in the context of reasoning with ambiguous expressions it seems to be adequate when we inspect the conjunctive and disjunctive desiderata as given in (2): it satisfies all of them. Consider the valuation V with $V(p) = 1$ and $V(q) = 0$. This is a model which falsifies $p \wedge q$, but does not falsify $?(p,q)$, and therefore $?(p,q) \not\models^2_{\mathsf{V}} p \wedge q$. The same model is a countermodel for $\neg p \wedge \neg q$, but not a countermodel for $\neg\,?(p,q)$, and so $\neg\,?(p,q) \not\models^2_{\mathsf{V}} \neg p \wedge \neg q$. We leave it to the reader to show that the other six desired properties of (2) hold for $\models^2_{\mathsf{V}}$.

3.2 An inference system for double barreled entailment

In [7] a normal Gentzen style deduction system has been presented for $\models^2_{\mathsf{V}}$.[7] In Figure 1 a sfsf-sequent system is given. The system is called **Q2**, where **Q** stands for Quasi logic, and **2** for the underlying double barreled entailment. By way of illustration, a derivation in **Q2** is given in Figure 2, where important **Q2**-sequents are highlighted in boxes.

A calculus for sfsf-sequents consists of three parts. As usual, there are structural rules and connective introduction rules. The third part consists of manipulation rules of the fsf arguments. Ambiguation rules, like AMBIG-LEFT and AMBIG-RIGHT are rules which introduce larger fsf's from smaller ones. Disambiguation rules, like DISAMB-MIXED, reduce the size of fsf's.

The rule DISAMB-MIXED looks somewhat bizarre. It will not be used in the systems in the following sections. In **Q2** it is needed to get a complete calculus. For example, $\emptyset \vdash_{\mathbf{Q2}} \emptyset$ is derivable with this rule.

$$\dfrac{\dfrac{\dfrac{}{\{p\} \vdash \{p\}}\ \text{START}}{\{p\},\{\neg p\} \vdash}\ \neg - \text{LIT-LEFT} \qquad \dfrac{\dfrac{}{\{q\} \vdash \{q\}}\ \text{START}}{\vdash \{\neg q\},\{q\}}\ \neg - \text{LIT-RIGHT}}{\emptyset \vdash \emptyset}\ \text{DISAMB-MIXED}$$

More important is the following observation about the disambiguation rule. It tells us that disambiguation can be replaced by two structural rules, which are not suited for practical use, but which are useful for establishing a completeness result (see appendix A).

OBSERVATION 9 A simple consequence of the disambiguation rule is that if $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ for some $\Gamma \in \mathfrak{X}$ and $\Delta \in \mathfrak{Y}$ then:

$$\frac{\mathfrak{X}, \Gamma' \vdash_{\mathbf{Q2}} \mathfrak{Y} \quad \mathfrak{X} \vdash_{\mathbf{Q2}} \Delta', \mathfrak{Y}}{\mathfrak{X} \vdash_{\mathbf{Q2}} \mathfrak{Y}}.$$

This observation can be generalized to a general structural rule for sfsf. We define first the *upward closure* $\overline{\mathfrak{X}}$ of an sfsf $\mathfrak{X}$ as being the sfsf which contains all finite supersets of formulae of the elements of $\mathfrak{X}$:

$$\overline{\mathfrak{X}} =_{df} \{\Gamma' \subseteq_f \mathcal{L} \mid \Gamma \subseteq \Gamma' \text{ for certain } \Gamma \in \mathfrak{X}\}.$$

This sfsf may be read as the collection of all further ambiguations of the ambiguous expressions represented by the elements of $\mathfrak{X}$. The disambiguation rule of **Q2** may now freely be replaced by:

(3) $$\frac{\mathfrak{X} \vdash \overline{\mathfrak{Y}} \quad \overline{\mathfrak{X}} \vdash \mathfrak{Y}}{\mathfrak{X} \vdash \mathfrak{Y}}.$$

Since all derivations are finite, it is not hard to show that $\mathfrak{X} \vdash_{\mathbf{Q2}} \mathfrak{Y}$ if and only if $\mathfrak{X}' \vdash_{\mathbf{Q2}} \mathfrak{Y}'$ for certain $\mathfrak{X}' \subseteq_f \mathfrak{X}, \mathfrak{Y}' \subseteq_f \mathfrak{Y}$. This finiteness property shows that the rule (3) is equivalent with DISAMB-MIXED.

A similar structural reinterpretation of the ambiguation rules is possible.

[7]In fact, the completeness claim for the calculus as given in [7] turned out to be wrong. Due to an oversight, the axiom $?(\neg\varphi, \neg\psi) \equiv_{\mathbf{Q2}} \neg\, ?(\varphi, \psi)$, which has been used in the completeness proof there, was missing. Simple addition of this axiom would immediately repair this deficit.

$$\mathfrak{X} \vdash \mathfrak{Y} \text{ if } \mathfrak{X} \cap \mathfrak{Y} \neq \emptyset \quad \textsc{start}$$

$$\frac{\mathfrak{X} \vdash \mathfrak{Y} \quad \mathfrak{X} \subseteq \mathfrak{X}' \quad \mathfrak{Y} \subseteq \mathfrak{Y}'}{\mathfrak{X}' \vdash \mathfrak{Y}'} \ \textsc{mon}$$

$$\frac{\mathfrak{X}, \Gamma \vdash \mathfrak{Y} \quad \mathfrak{X}' \vdash \Gamma, \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}' \vdash \mathfrak{Y}, \mathfrak{Y}'} \ \textsc{cut}$$

$$\frac{\mathfrak{X} \vdash \mathfrak{Y}}{\neg\mathfrak{Y} \vdash \neg\mathfrak{X}} \ \textsc{contrap}$$

$$\frac{\mathfrak{X}, \Gamma, \Delta \vdash \mathfrak{Y} \quad \mathfrak{X}' \vdash \Gamma', \Delta', \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}', \Gamma \cap \Delta \vdash \Gamma' \cap \Delta', \mathfrak{Y}, \mathfrak{Y}'} \ \textsc{disamb-mixed}$$

$$\frac{\mathfrak{X}, \Gamma \vdash \mathfrak{Y} \quad \mathfrak{X}', \Delta \vdash \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}', \Gamma \cup \Delta \vdash \mathfrak{Y}, \mathfrak{Y}'} \ \textsc{ambig-left}$$

$$\frac{\mathfrak{X} \vdash \Gamma, \mathfrak{Y} \quad \mathfrak{X}' \vdash \Delta, \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}' \vdash \Gamma \cup \Delta, \mathfrak{Y}, \mathfrak{Y}'} \ \textsc{ambig-right}$$

$$\frac{\mathfrak{X}, \Gamma \cdot \varphi \vdash \mathfrak{Y} \quad \mathfrak{X}' \vdash \psi \cdot \Delta, \mathfrak{Y}}{\mathfrak{X}, \mathfrak{X}', \Gamma \cdot \neg\psi \vdash \neg\varphi \cdot \Delta, \mathfrak{Y}, \mathfrak{Y}'} \ \neg\textsc{-mixed}$$

$$\frac{\mathfrak{X} \vdash P, \mathfrak{Y} \quad P \subseteq I\!\!P \cup \neg I\!\!P}{\mathfrak{X}, \neg P \vdash \mathfrak{Y}} \ \neg\textsc{-lit-left}$$

$$\frac{\mathfrak{X}, P \vdash \mathfrak{Y} \quad P \subseteq I\!\!P \cup \neg I\!\!P}{\mathfrak{X} \vdash \neg P, \mathfrak{Y}} \ \neg\textsc{-lit-right}$$

$$\frac{\mathfrak{X}, \Gamma \vdash \mathfrak{Y}}{\mathfrak{X}, \neg\neg\Gamma \vdash \mathfrak{Y}} \ \neg\neg\textsc{-left}$$

$$\frac{\mathfrak{X} \vdash \Gamma, \mathfrak{Y}}{\mathfrak{X} \vdash \neg\neg\Gamma, \mathfrak{Y}} \ \neg\neg\textsc{-right}$$

$$\frac{\mathfrak{X}, \Gamma, \Delta \vdash \mathfrak{Y}}{\mathfrak{X}, \Gamma \wedge \Delta \vdash \mathfrak{Y}} \ \wedge\textsc{-left}$$

$$\frac{\mathfrak{X} \vdash \Gamma, \Delta \quad \mathfrak{X}' \vdash \Delta, \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}' \vdash \Gamma \wedge \Delta, \mathfrak{Y}, \mathfrak{Y}'} \ \wedge\textsc{-right}$$

$$\frac{\mathfrak{X}, \Gamma \cdot \varphi \cdot \psi \vdash \mathfrak{Y}}{\mathfrak{X}, \Gamma \cdot ?(\varphi, \psi) \vdash \mathfrak{Y}} \ ?\textsc{-left}$$

$$\frac{\mathfrak{X} \vdash \varphi \cdot \psi \cdot \Gamma, \mathfrak{Y}}{\mathfrak{X} \vdash ?(\varphi, \psi) \cdot \Gamma, \mathfrak{Y}} \ ?\textsc{-right}$$

Figure 1: The system **Q2**

1.	$\{\varphi\} \vdash_{\mathbf{Q2}} \{\varphi\}$	START	
2.	$\{\psi\} \vdash_{\mathbf{Q2}} \{\psi\}$	START	
3.	$\{\varphi, \psi\} \vdash_{\mathbf{Q2}} \{\varphi\}, \{\psi\}$	AMBIG-LEFT	1,2
4.	$\boxed{\{?(\varphi, \psi)\} \vdash_{\mathbf{Q2}} \{\varphi\}, \{\psi\}}$	?-LEFT	3
5.	$\{\varphi, \neg\varphi\} \vdash_{\mathbf{Q2}} \{\psi, \neg\psi\}$	¬-MIXED	1,2
6.	$\boxed{\{?(\varphi, \neg\varphi)\} \vdash_{\mathbf{Q2}} \{?(\psi, \neg\psi)\}}$	?-rules	5
7.	$\{\varphi, \psi\} \vdash_{\mathbf{Q2}} \{\varphi, \psi\}$	START	
8.	$\{?(\varphi, \psi)\} \vdash_{\mathbf{Q2}} \{\varphi, \psi\}$	?-LEFT	7
9.	$\{\neg\varphi, \neg\psi\} \vdash_{\mathbf{Q2}} \{\neg\, ?(\varphi, \psi)\}$	CONTRAP	8
10.	$\boxed{\{?(\neg\varphi, \neg\psi)\} \vdash_{\mathbf{Q2}} \{\neg\, ?(\varphi, \psi)\}}$	?-LEFT	9
11.	$\{?(\varphi, \neg\varphi)\} \vdash_{\mathbf{Q2}} \{?(\neg\varphi, \neg\neg\varphi)\}$	*Subst.*$[\varphi/\varphi, \psi/\neg\varphi]$	6
12.	$\{?(\neg\varphi, \neg\neg\varphi)\} \vdash_{\mathbf{Q2}} \{\neg\, ?(\varphi, \neg\varphi)\}$	*Subst.*$[\varphi/\varphi, \psi/\neg\varphi]$	9
13.	$\boxed{\{?(\varphi, \neg\varphi)\} \vdash_{\mathbf{Q2}} \{\neg\, ?(\varphi, \neg\varphi)\}}$	CUT	11,12
14.	$\{\varphi\}, \{\psi, \chi\} \vdash_{\mathbf{Q2}} \{\varphi\}$	START	
15.	$\{\varphi\}, \{\psi, \chi\} \vdash_{\mathbf{Q2}} \{\psi, \chi\}$	START	
16.	$\{\varphi\}, \{\psi, \chi\} \vdash_{\mathbf{Q2}} \{\varphi \wedge \psi, \varphi \wedge \chi\}$	∧-RIGHT	14,15
17.	$\{\varphi\}, \{?(\psi, \chi)\} \vdash_{\mathbf{Q2}} \{?(\varphi \wedge \psi, \varphi \wedge \chi)\}$	?-rules	16
18.	$\boxed{\{\varphi \wedge ?(\psi, \chi)\} \vdash_{\mathbf{Q2}} \{?(\varphi \wedge \psi, \varphi \wedge \chi)\}}$	∧-LEFT	17

Figure 2: A derivation in **Q2**

OBSERVATION 10 The ambiguation rules can be replaced as by the following structural variants:

$$\frac{\mathfrak{X} \vdash \mathfrak{Y}}{\overline{\mathfrak{X}} \vdash \mathfrak{Y}} \text{ and } \frac{\mathfrak{X} \vdash \mathfrak{Y}}{\mathfrak{X} \vdash \overline{\mathfrak{Y}}}.$$

THEOREM 11 (SOUNDNESS AND COMPLETENESS OF **Q2**) *For every pair of sfsf's $\mathfrak{X}, \mathfrak{Y}$:*

$$\mathfrak{X} \models^2_{\mathsf{V}} \mathfrak{Y} \Longleftrightarrow \mathfrak{X} \vdash_{\mathbf{Q2}} \mathfrak{Y}.$$

Proof. See appendix A. $\triangleleft$

3.3 Evaluation

The double-barreled logic seems suitable at first sight, that is, after the first inspection of its logical interplay with disjunction and conjunction. These criteria always play the role of the first most elementary test for reasoning with ambiguities, since a pure disjunctive interpretation of ambiguous expressions leads to counterintuitive results. The negation of an ambiguous expression, which has two readings of which one is weaker than the other, entails the negation of the weakest reading, which is stronger than the negation of the strongest reading [5]. The proper conclusion would be the negation of the strongest reading, since this is the only safe conclusion which we may draw.

This is what we also may expect from the double-barreled logic. These kind of safe conclusions hold indeed for our double-barreled logic.

$$\varphi \models^2_{\mathsf{V}} \psi \implies ?(\varphi,\psi) \models^2_{\mathsf{V}} \psi \text{ and } \neg\, ?(\varphi,\psi) \models^2_{\mathsf{V}} \neg\varphi$$

Moreover, entailment of the stronger readings is not sound in this double-barreled logic.

$$\varphi \models^2_{\mathsf{V}} \psi \not\Rightarrow ?(\varphi,\psi) \models^2_{\mathsf{V}} \varphi \text{ and } \varphi \models^2_{\mathsf{V}} \psi \not\Rightarrow \neg\, ?(\varphi,\psi) \models^2_{\mathsf{V}} \neg\psi$$

The first property does not come as a surprise. The second reflects that $\neg\, ?(\varphi,\psi) \not\models^2_{\mathsf{V}} \neg(\varphi\vee\psi)$. By way of illustration, consider the case of a simple example of scope ambiguity of a negation and a conjunction: $?(\neg(p\wedge q), \neg p\wedge q)$. The first element is weaker than the second: $\neg p\wedge q \models^2_{\mathsf{V}} \neg(p\wedge q)$, nevertheless $\neg\, ?(\neg(p\wedge q), \neg p\wedge q) \not\models^2_{\mathsf{V}} \neg\neg(p\wedge q)$. Since not every model which falsifies $\neg\neg(p\wedge q)$ does not necessarily falsify $\neg\, ?(\neg(p\wedge q), \neg p\wedge q)$ (take the valuation V such that $V(p) = 0$ and $V(q) = 1$).

Classical logic still holds for the ?-free language $\mathcal{L}_0$. Weakening comes with partiality, in this setting with formulas containing ?. Just like in ordinary partial logic, this weakening mainly consists of rejection of rules for negation. In general the classical rules $\neg$-LEFT and $\neg$-RIGHT do not hold.

(4) $$\varphi\wedge\neg\varphi \not\models^2_{\mathsf{V}} \bot \text{ and } \top \not\models^2_{\mathsf{V}} \varphi\vee\neg\varphi.$$

The latter is a standard absence of three valued logic, the former is caused by the double barreled requirement on deriving conclusions: $\varphi\wedge\neg\varphi$ is not always false.

The pay off of enforcing the disjunctive and conjunctive desiderata of (2) by using the double barreled requirement on consequences is that we end up with the following principle:

(5) $$\varphi\wedge\neg\varphi \models^2_{\mathsf{V}} \psi\vee\neg\psi.$$

In fact, this is the way to express the excluded fourth value in a double barreled setting [27].[8] The general principle in a sequential setting is reflected by the rule $\neg$-MIXED present in the system **Q2**. The rule itself looks harmless, but there seems to be no way to motivate it in the absence of the general exclusion rules as given in (4).

More essential inadequacies of the double barreled logic are caused by the maximal evaluation of ambiguous expressions, that is, all the readings are taken into account when it comes to truth-value assignment. This means that readings cannot be excluded, even if we know that they cannot be true on the basis of additional information. In the double barreled logic this amounts to the following striking failure:

(6) $$\neg\varphi, ?(\varphi,\psi) \not\models^2_{\mathsf{V}} \psi.$$

Note that $\psi \not=\!\!\!|_{\mathsf{V}} \neg\varphi, ?(\varphi,\psi)$; if φ is undefined and ψ is false then the left hand side is false, but none of the arguments on the right hand side is false.

The failure in (6) may suggest that our logic is too weak. Yet in another way, interpretation of ambiguity in a maximal way also leads to too strong conclusions. An expression with

[8]In a normal single barreled setting the rule $\neg -$ LEFT would be the proper exclusion of the fourth value.

mutually contradictory readings is always undefined. Since it has no models and no counter-models it entails any other expression with mutually incompatible readings. This becomes most clear from the following entailment:

(7) $$?(\varphi, \neg\varphi) \models^2_{\mathsf{V}} ?(\psi, \neg\psi).$$

This entailment seems highly undesirable. There is no argument to exclude such mutually contradictory reading sets. Ambiguities of reference and lexical ambiguities give rise to such incompatible resolutions in a very natural way, and we surely do not wish to assign such expressions such a logically flat structure which yields utterly strange entailments such as in (7).[9]

Yet another problem of 'maximal evaluation' is that ambiguity is logically transparent. All connectives can be freely distributed over the readings of an ambiguous expression:

(8) $$\begin{array}{rcl@{\qquad}rcl} \neg\,?(\varphi,\psi) & \models^2_{\mathsf{V}} & ?(\neg\varphi,\neg\psi) & ?(\neg\varphi,\neg\psi) & \models^2_{\mathsf{V}} & \neg\,?(\varphi,\psi) \\ \varphi\wedge ?(\psi,\chi) & \models^2_{\mathsf{V}} & ?(\varphi\wedge\psi,\varphi\wedge\chi) & ?(\varphi\wedge\psi,\varphi\wedge\chi) & \models^2_{\mathsf{V}} & \varphi\wedge ?(\psi,\chi) \\ ?(\varphi,?(\psi,\chi)) & \models^2_{\mathsf{V}} & ?(?(\varphi,\psi),?(\varphi,\chi)) & ?(?(\varphi,\psi),?(\varphi,\chi)) & \models^2_{\mathsf{V}} & ?(?(\varphi,\psi),?(\varphi,\chi)). \end{array}$$

Only the last distributions of ?'s seems safe. The others are not always wanted, since we may wish to disambiguate a subformula also on the basis of where it appears [25]. For example, we do not want to identify $\neg\varphi\wedge ?(\varphi,\psi)$ with $?(\neg\varphi\wedge\varphi, \neg\varphi\wedge\psi)$, but rather with $\neg\varphi\wedge\psi$. In order to establish such reasoning patterns the maximal evaluation has to be given up, and we have to take the possibility into account that readings of ambiguous expressions may be excluded.

4 Relative ambiguity

The most simple setting of relative evaluation is obtained by attaching a restriction on readings to ordinary truth-value assignments. Such restrictions are called partial disambiguations.

DEFINITION 12 *A partial disambiguation is a function* $\mathsf{a} : \wp_f\mathcal{L} \longrightarrow \wp_f\mathcal{L}_0$ *such that* $\mathsf{a}(\Gamma) = \emptyset$ *iff* $\Gamma = \emptyset$, $\mathsf{a}(\Delta) \subseteq \mathsf{Res}(\Delta)$ *and* $\mathsf{a}(\Delta \cup ?(\varphi,\psi)) = \mathsf{a}(\Delta \cup \{\varphi,\psi\})$ *for all fsf's* Γ, Δ *and formulas* φ, ψ.

Note that $\mathsf{a}(\{\varphi\}) = \{\varphi\}$ for all $\varphi \in \mathcal{L}_0$.

The appropriate class of models are pairs of valuations and partial disambiguation. Truth-value assignment is then defined with respect to the limited set which is given by this partial disambiguation.

DEFINITION 13 *A disambiguation model is a pair* $M = \langle V, \mathsf{a}\rangle$ *with* $V \in \mathsf{V}$ *and* a *a partial disambiguation. The class of disambiguation models is called* D.

An fsf Γ *is* true *at a model* $M = \langle V, \mathsf{a}\rangle$ *if* $V \models \mathsf{a}(\Gamma)$, *an fsf is* false *at* M *if* $V \dashv \mathsf{a}(\Gamma)$.

[9] To illustrate this effect in an even more dramatic manner, note that also $?(\varphi,\neg\varphi) \models^2_{\mathsf{V}} \neg\,?(\varphi,\neg\varphi)$.

4.1 A short evaluation

Over the class of disambiguation models the double barreled notion of entailment is not needed to obtain the disjunctive and conjunctive desiderata as represented in (2). All the requirements hold for the single barreled relation $\models_{\mathsf{D}}$ on the assumption that φ and ψ in $?(\varphi, \psi)$ are both ?-free.

OBSERVATION 14 In general we have that if $\mathsf{Res}(\Gamma) = \{\alpha_1, \ldots, \alpha_n\}$ then

(9) $$\Gamma \models_{\mathsf{D}} \{\alpha_1\}, \ldots, \{\alpha_n\} \text{ but } \Gamma \not\models_{\mathsf{D}} \{\alpha_1, \alpha_2\}, \{\alpha_3\}, \ldots, \{\alpha_n\}.$$

In particular, $?(\varphi, \psi) \models_{\mathsf{D}} \varphi \vee \psi$ if $\varphi, \psi \in \mathcal{L}_0$. This does not hold if $\varphi \notin \mathcal{L}_0$, e.g., $?(?(p,q),r) \not\models_{\mathsf{D}} ?(p,q) \vee r$. We may take $V(p) = 1$ and $V(q) = V(r) = 0$ and $\mathsf{a}(\{p,q,r\}) = \{p\}$ and $\mathsf{a}(\{p,q\}) = \{p,q\}$, then $\langle V, \mathsf{a}\rangle \models ?(?(p,q),r)$ but $\langle V, \mathsf{a}\rangle \not\models ?(p,q) \vee r$. In words, an ambiguous expression entails the disjunction of the set of its full resolutions, but not the disjunction of a complete set of partial disambiguations.

These observations show that the general AMBIG-rules are unsound with respect to this general semantics. In section 4.3 we will show how to restrain partial disambiguations in order to restore these rules.

The possibility of excluding readings of ambiguous expressions establishes satisfiability for expressions with mutually contradictory readings. Indeed, we get the opposite results of (6) and (7):

(10) $$\neg\varphi, \{\varphi, \psi\} \models_{\mathsf{D}} \psi \text{ and } \{\varphi, \neg\varphi\} \not\models_{\mathsf{D}} \{\psi, \neg\psi\}$$

Furthermore, the logical transparency of ? disappears. The operator ? no longer distributes over the ordinary propositional connectives (and vice versa). Only distribution of ? over ? is still correct. We do not want a logical difference between $?(\varphi, \psi)$ and the set $\{\varphi, \psi\}$. Since, this transparency does not hold for negation and conjunction, a calculus for this entailment has to restrain the negation and conjunction rules such as in **Q2** to ?-free singletons.

Moreover, none of the exclusion principles hold:

(11) $$\varphi \wedge \neg\varphi \not\models_{\mathsf{D}} \bot\ ,\ \top \not\models_{\mathsf{D}} \varphi \vee \neg\varphi \text{ and } \varphi \wedge \neg\varphi \not\models_{\mathsf{D}} \psi \vee \neg\psi.$$

As in **Q2**, the first two principles hold for ?-free propositions. Classical logic for the standard sublanguage remains. The difference with **Q2** is the failure of the third principle, which is a mere side-effect of the double barreled definition of entailment in the previous section.

The basic reason of the unsoundness of the exclusion principles is that a partial disambiguation of a proposition and its negation not necessarily leads to a contradiction.

4.2 Calculi of relative ambiguity

DEFINITION 15 **Q** *is the sfsf-sequent system with all the standard structural rules,* START, MON *and* CUT, *the classical rules for the connectives restricted to $\mathcal{L}_0$-singletons (see Figure 3), the rules* ?-LEFT *and* ?-RIGHT *(see Figure 1) and the following minimal ambiguation rules with* $\mathsf{Res}(\Gamma) = \{\varphi_1, ..., \varphi_n\}$*:*

$$\frac{\mathfrak{X}_1, \varphi_1 \vdash \mathfrak{Y}_1 \quad \ldots \quad \mathfrak{X}_n, \varphi_n \vdash \mathfrak{Y}_n}{\mathfrak{X}_1, \ldots, \mathfrak{X}_n, \Gamma \vdash \mathfrak{Y}_1, \ldots, \mathfrak{Y}_n} \quad \text{AMBIG-SINGLES-LEFT}$$

$$\frac{\mathfrak{X}_1 \vdash \varphi_1, \mathfrak{Y}_1 \quad \ldots \quad \mathfrak{X}_n \vdash \varphi_n, \mathfrak{Y}_n}{\mathfrak{X}_1, \ldots, \mathfrak{X}_n \vdash \Gamma, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_n} \quad \text{AMBIG-SINGLES-RIGHT}$$

$\dfrac{\mathfrak{X} \vdash \alpha, \mathfrak{Y}}{\mathfrak{X}, \neg\alpha \vdash \mathfrak{Y}}$	$\neg$ – SNG-$\mathcal{L}_0$-LEFT	$\dfrac{\mathfrak{X}, \alpha \vdash \mathfrak{Y}}{\mathfrak{X} \vdash \neg\alpha, \mathfrak{Y}}$	$\neg$ – SNG-$\mathcal{L}_0$-RIGHT
$\dfrac{\mathfrak{X}, \alpha, \beta \vdash \mathfrak{Y}}{\mathfrak{X}, \alpha \wedge \beta \vdash \mathfrak{Y}}$	$\wedge$ – SNG-$\mathcal{L}_0$-LEFT	$\dfrac{\mathfrak{X} \vdash \alpha, \mathfrak{Y} \quad \mathfrak{X}' \vdash \beta, \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}' \vdash \alpha \wedge \beta, \mathfrak{Y}, \mathfrak{Y}'}$	$\wedge$ – SNG-$\mathcal{L}_0$-RIGHT

Figure 3: The classical rules for $\mathcal{L}_0$ singleton. Formulas α and β are ?-free.

THEOREM 16 (SOUNDNESS AND COMPLETENESS OF $\mathbf{Q}$) *For every pair $\mathfrak{X}, \mathfrak{Y}$ of sfsf's*

$$\mathfrak{X} \models_{\mathsf{D}} \mathfrak{Y} \Longleftrightarrow \mathfrak{X} \vdash_{\mathbf{Q}} \mathfrak{Y}.$$

Proof. Soundness of propositional logic for singletons follows from the fact that $\mathsf{a}(\{\varphi\}) = \{\varphi\}$ for all $\varphi \in \mathcal{L}_0$. The soundness of the ambiguation rules follow from $\mathsf{a}(\Gamma) \subseteq \mathsf{Res}(\Gamma)$ and $\mathsf{a}(\Gamma) \neq \emptyset$ for all finite non-empty sets Γ. The proof of completeness can be found in appendix B. $\lhd$

4.3 Monotonicity for partial disambiguations

The failure of soundness for general AMBIG-rules is a simple consequence of the fact that partial disambiguation may behave in a non-monotonic fashion. This has been made clear in the examples in paragraph 4.1 (Observation 14), e.g., $\{p,q,r\} \not\models_{\mathsf{D}} \{p,q\}, r$. An ambiguous expression does not necessarily entail the disjunction of a complete set of partial readings. Moreover, the conjunction of a complete set of partial readings an ambiguous expression does not entail the expression itself, e.g., $\{p,q\}, r \not\models_{\mathsf{D}} \{p,q,r\}$.

Left-monotonic disambiguations In order to reintroduce the full AMBIG-rules we need to restrict the class of partial disambiguations. These restrictions are a kind of monotonicity requirements. The first monotonicity requirement which corresponds to rehabilitation of LEFT-AMBIG can be described in the following characteristic manner.

DEFINITION 17 *A partial disambiguation* a *is called* left-monotonic *if for all fsf's* Γ, Δ*:*

$$\mathsf{a}(\Gamma) \subseteq \mathsf{a}(\Gamma \cup \Delta) \text{ or } \mathsf{a}(\Delta) \subseteq \mathsf{a}(\Gamma \cup \Delta).$$

The class of disambiguation models with a left-monotonic partial disambiguation is called Dl.

The partial disambiguation as given in Observation 14 violates this requirement, since $\mathsf{a}(\{r\}) = \{r\} \not\subseteq \mathsf{a}(\{p,q,r\})$ and $\mathsf{a}(\{p,q\}) \not\subseteq \mathsf{a}(\{p,q,r\})$.

Left-monotonicity captures the intuition that, since Γ and Δ represent two partial readings of $\Gamma \cup \Delta$, it should be the case that the disambiguation of at least one of them should count as a further disambiguation of $\Gamma \cup \Delta$. This represents a monotonicity criterion for partial disambiguations, because otherwise partial disambiguation brings us no further in specifying underspecified expressions.

It would be too strong a demand on disambiguations that both disambiguations $\mathsf{a}(\Gamma)$ and $\mathsf{a}(\Delta)$ should be more specific than the disambiguation $\mathsf{a}(\Gamma \cup \Delta)$ of the original expression itself. It may be the case that $\mathsf{a}(\Gamma \cup \Delta) \subseteq \Delta$, excluding the partial reading Γ. Γ may represent a hopelessly ambiguous expression which gets excluded by the information given by a. Requiring full monotonicity — $\mathsf{a}(\Gamma) \subseteq \mathsf{a}(\Gamma \cup \Delta)$ *and* $\mathsf{a}(\Delta) \subseteq \mathsf{a}(\Gamma \cup \Delta)$[10] — is also undesirable for technical reasons. It restores maximal interpretation of ambiguous expression:

$$\text{If } \Gamma \subseteq \mathcal{L}_0 \text{ then } \mathsf{a}(\Gamma) = \bigcup_{\varphi \in \Gamma} \mathsf{a}(\varphi) = \bigcup_{\varphi \in \Gamma} \varphi = \Gamma.$$

The left-monotonicity constraint on partial disambiguation restores LEFT-AMBIG:

$$\mathfrak{X}, \Gamma \models_{\mathsf{Dl}} \mathfrak{Y} \;\&\; \mathfrak{X}', \Delta \models_{\mathsf{Dl}} \mathfrak{Y}' \Rightarrow \mathfrak{X}, \mathfrak{X}', \Gamma \cup \Delta \models_{\mathsf{Dl}} \mathfrak{Y}, \mathfrak{Y}'.$$

The most obvious and important consequence of this entailment is that $\Gamma \cup \Delta \models_{\mathsf{Dl}} \Gamma, \Delta$. More general,

$$\Gamma \models_{\mathsf{Dl}} \Gamma_1, \ldots, \Gamma_n \text{ if } \bigcup \Gamma_i = \Gamma.$$

Moreover, left-monotonicity precisely characterizes the logic **Ql** which consists of the rules of **Q** with AMBIG-SINGLES-LEFT replaced by the AMBIG-LEFT of **Q2**.

THEOREM 18 (SOUNDNESS AND COMPLETENESS OF **Ql**) *For all sfsf's $\mathfrak{X}, \mathfrak{Y}$:*

$$\mathfrak{X} \models_{\mathsf{Dl}} \mathfrak{Y} \iff \mathfrak{X} \vdash_{\mathbf{Ql}} \mathfrak{Y}.$$

Proof. See appendix B. $\lhd$

Right-monotonic disambiguations The full ambiguation rule AMBIG-RIGHT is also unsound for the class of left-monotonic disambiguation models. Let a be a left-monotonic partial disambiguation with $\mathsf{a}(\{p,q,r\}) = \{p,q,r\}$ and $\mathsf{a}(\{p,q\}) = \{p\}$. Let V be the valuation such that $V(p) = V(r) = 1$ and $V(q) = 0$, then $\langle V, \mathsf{a}\rangle \models \{p,q\}$, $\langle V, \mathsf{a}\rangle \models \{r\}$ but $\langle V, \mathsf{a}\rangle \not\models \{p,q,r\}$. Therefore, $\{p,q\}, r \not\models_{\mathsf{Dl}} \{p,q,r\}$.

In order to get AMBIG-RIGHT back, we need to impose the following monotonicity constraint on partial disambiguations.

DEFINITION 19 *A partial disambiguation* a *is called right-monotonic if for all fsf's Γ and Δ:*

$$\mathsf{a}(\Gamma \cup \Delta) \subseteq \mathsf{a}(\Gamma) \cup \mathsf{a}(\Delta).$$

The class of disambiguation models with an associated right-monotonic partial disambiguation is called Dr. *The class of disambiguation models with partial disambiguations which are both left- and right-monotonic is called* Dlr.

The right-monotonicity requirement is a 'sum–is–more–than–parts' principle, that is, the disambiguations of a pair of partial readings contain less (or the same) information as the disambiguations of the original expression.

The system **Qr** consists of the **Q**-rules with AMBIG-SINGLES-RIGHT replaced by AMBIG-RIGHT. The system **Qlr** consists of the **Q**-rules with the two ambiguation rules for singletons replaced by the full ambiguation rules of **Q2**.

[10] Note that this requirement is equivalent to the more standard description of full monotonocity: $\Gamma \subseteq \Delta \Rightarrow \mathsf{a}(\Gamma) \subseteq \mathsf{a}(\Delta)$ for all fsf's Γ and Δ.

THEOREM 20 (SOUNDNESS AND COMPLETENESS OF **Qr** AND **Qlr**) *For all sfsf's $\mathfrak{X}, \mathfrak{Y}$:*

$$\mathfrak{X} \models_{\mathsf{Dr}} \mathfrak{Y} \iff \mathfrak{X} \vdash_{\mathbf{Qr}} \mathfrak{Y} \text{ and}$$
$$\mathfrak{X} \models_{\mathsf{Dlr}} \mathfrak{Y} \iff \mathfrak{X} \vdash_{\mathbf{Qlr}} \mathfrak{Y}$$

Proof. The reader may check for himself that the following reasoning pattern hold for entailment over the class of right-monotonic disambiguation models:

$$\Gamma_1, \ldots, \Gamma_n \models_{\mathsf{Dr}} \Gamma \text{ if } \bigcup_{i=1}^{n} \Gamma_i = \Gamma.$$

The following generalization holds for the class Dlr:

$$\Gamma_1, \ldots, \Gamma_n \models_{\mathsf{Dlr}} \Delta_1, \ldots, \Delta_m \text{ if } \bigcup_{i=1}^{n} \Gamma_i = \bigcup_{j=1}^{m} \Delta_j.$$

These entailments are the main consequences of the general rule schemes of RIGHT-AMBIG and LEFT&RIGHT-AMBIG respectively. Full soundness for those rules can easily obtained from the entailments above. The completeness proof can be found in appendix B. $\triangleleft$

4.4 Supplying propositional logic to disambiguation

The attractive property of relative evaluation with respect to partial disambiguation is that it gives us a very minimal logic, and that it is very simple to find corresponding constraints on partial disambiguations for additional rules which one possibly would like to implement for reasoning with ambiguous expressions. The monotonicity constraints that correspond to the AMBIG-rules are examples of this freedom. It may be the case that one also wishes other principles restored. One may have a certain class of ambiguity phenomena in mind for which some logical transparency of expressions is needed.

For example, if we want a logic which consists of **Ql** and a classical $\neg$-LEFT rule, then the corresponding class of disambiguation models $\langle V, \mathsf{a} \rangle$ are all models in Dl such that $\mathsf{a}(\neg\Gamma) \cap \neg\, \mathsf{a}(\Gamma) \neq \emptyset$ for all non-empty fsf's Γ.

OBSERVATION 21 The system which consists of the rules **Ql** and the general $\neg$-LEFT:

$$\frac{\mathfrak{X} \vdash \Gamma, \mathfrak{Y}}{\mathfrak{X}, \neg\Gamma \vdash \mathfrak{Y}}.$$

is sound and complete with respect to the class of models mentioned above.

Proof. Let M be the corresponding subclass of Dl. Suppose that $\mathfrak{X} \models_{\mathsf{M}} \Gamma, \mathfrak{Y}$ (*) and $\langle V, \mathsf{a} \rangle \in \mathsf{M}$ such that $\langle V, \mathsf{a} \rangle \models \Delta$ for all fsf's $\Delta \in \mathfrak{X} \cup \{\neg\Gamma\}$. This means that $V \models \mathsf{a}(\neg\Gamma)$. Since $\mathsf{a}(\neg\Gamma) \cap \neg\, \mathsf{a}(\Gamma) \neq \emptyset$, there exists $\varphi \in \mathsf{a}(\Gamma)$ such that $\neg\varphi \in \mathsf{a}(\neg\Gamma)$, and therefore, $V \dashv \varphi$. From the last conclusion, and the fact that $\varphi \in \mathsf{a}(\Gamma)$ we have $V \not\models \Gamma$. The assumption (*) subsequently enforces the existence of a fsf $\Theta \in \mathfrak{Y}$ such that $\langle V, \mathsf{a} \rangle \models \Theta$. Because $\langle V, \mathsf{a} \rangle \in \mathsf{M}$ was chosen arbitrarily, we know that $\mathfrak{X}, \neg\Gamma \models_{\mathsf{M}} \mathfrak{Y}$. In other words,

$$\mathfrak{X} \models_{\mathsf{M}} \Gamma, \mathfrak{Y} \Rightarrow \mathfrak{X}, \neg\Gamma \models_{\mathsf{M}} \mathfrak{Y}.$$

The completeness proof can be found in appendix B. $\lhd$

To restore the full $\wedge$-rules we need to postulate that $\mathsf{a}(\Gamma \wedge \Delta) \subseteq \mathsf{a}(\Gamma) \wedge \mathsf{a}(\Delta)$ for $\wedge$-RIGHT and the converse for $\wedge$-LEFT. Similar constraints would repair the full double negation rules, while a general version of $\neg$-RIGHT requires total disambiguation with transparency for $\neg$: $\mathsf{a}(\Gamma) = \{\alpha_\Gamma\}$ and $\mathsf{a}(\neg\Gamma) = \{\neg\alpha_\Gamma\}$ for certain $\alpha_\Gamma \in \mathsf{Res}(\Gamma)$.

5 Conclusions and reflections

In this paper we have shown that through relating truth-conditions of ambiguous expressions to disambiguations it is possible to obtain decent Gentzen calculi for reasoning with this type of uncertainty. Moreover, the resulting calculus **Q** can be used as a minimal calculus for logics of ambiguous expressions. We have shown in the last paragraph how additional rules can be enforced by simple characteristic structural constraints on partial disambiguations. Since we have made no commitments to any kind of representation or resolution method of ambiguous expressions, such a high degree of modularity offers a safe logical basis for starting investigations of the logic of specific kinds of ambiguity and existing representation and resolution techniques.

5.1 Further structuring of disambiguations, and the near future

An objection against our neutral logical attitude towards representation and resolution is that the inevitable lack of expressivity leads to the indescernability of expressions that share the same set of full resolutions. A straightforward modification would be to use additional indices to make this distinction possible. Such an index is then used to mark an ambiguous expression, and appears in the logic as a kind of formula variable (often called propositional meta-variables in the context of representation of ambiguous expressions). In such a setting partial disambiguations are mappings from pairs of indices and fsf's to ?-free fsf's. The corresponding AMBIG-rules would then be assigned the following format:

$$\frac{\mathfrak{X}, \Gamma_x \vdash \mathfrak{Y} \quad \mathfrak{X}', \Delta_x \vdash \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}', (\Gamma \cup \Delta)_x \vdash \mathfrak{Y}, \mathfrak{Y}'}\ \text{IND-AMBIG-LEFT} \qquad \frac{\mathfrak{X} \vdash \Gamma_x, \mathfrak{Y} \quad \mathfrak{X}' \vdash \Delta_x, \mathfrak{Y}'}{\mathfrak{X}, \mathfrak{X}' \vdash (\Gamma \cup \Delta)_x, \mathfrak{Y}, \mathfrak{Y}'}\ \text{IND-AMBIG-RIGHT}.$$

Of course, similar monotonicity criteria as for the logics **Ql** and **Qr** have to be used for indexed partial disambiguations to guarantee the soundness of the rules above (but then per index).

Another objection which we like to mention here is that the relativity of ambiguity which we used in this paper is not sufficient. Further relativity can be imposed by extending the elimination potential of disambiguations. Partial disambiguations may exclude possible readings, but it may also exclude possible further disambiguations. For example, we may exclude r from a reading set $\{p, q, \neg q, r\}$, but we also know that some subsets of the readings are incompatible. In this paper, the propositional language which we used is simply not rich enough to capture such 'intensionally' exclusive information of disambiguations, but even for our general approach it seems to be of main interest to our research to start investigation of languages which make it possible to reason about 'further disambiguation'. This leads us in the direction of Pinkal's so-called precisification semantics [23], where information orders over partial truth-value assignments are used to model intensional reasoning about disambiguations. From a logical perspective, such a semantic structure points in the direction of possible world semantics for constructive logics, such as intuitionistic logic [17] and — closer to partial logic — Nelson's logic of constructible falsity [21, 13]. Constructive implication would then facilitate reasoning about further specification of ambiguous expressions. The truth of an expression $\varphi \to \psi$ means that every possible further disambiguation of the current information which yields verification of φ also establishes verification of ψ.[11] This challenge, in combination with the incorporation of formula variables which we mentioned above, is the main subject of our current research.

5.2 And in the long term ...

As said in the introduction, this paper contains just our first careful steps from logic into the direction of 'real' representation systems of underspecification. It remains to be seen how different peculiarities of different systems can be interpreted and axiomatized. This open ending was the main motivation to keep the level of expressiveness as low as possible.

A first step would be to introduce formula variables as indicated in the previous paragraph, and generalize this to facilitate variation over other types then formulas only. In this way it will be easier to find our way in the direction of axiomatizing first-order 'quasi logics' [2],[12] since then ambiguity on individuals, properties and relations come into the picture.

In line with the intensional modelling of ambiguity mentioned in the previous paragraph, we think of dynamic and nonmonotonic extensions as promising future challenges. Such 'disambiguation order models' create a lot of possibilities in this direction. Having a proper notion of information states in hand, dynamification is conceivable [16]. Nonmonotonic 'possibility' operators over information ordered models also have been put into a general logical perspective [12, 29, 30].[13] Combinations of such logics would bring the context-dependence and defeasibility of reasoning with ambiguities into focus [15].

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[11] Also modal reasoning about 'further disambiguation' seems to be an appropriate extension, which leads into the direction as chosen by Fernando in [10].

[12] For a first-order version of the double barreled logic **Q2** see [7]. In [18] the reader finds a first-order tableaux system defined on the basis of the double barreled entailment over full resolution sets.

[13] Another possibility of enabling deafisible reasoning about ambiguous expressions would be to take the proof-theoretic view of default logic on information orders [24]. A second syntactic alternative is to build in non-monotonicity is to presume preferentially structured reading sets of ambiguous expressions in line with [9].

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Appendix A: The completeness of Q2

In the appendices we omit the soundness proofs of the different systems. They can be proved by straightforward induction on the length of derivations.

Saturated sets As in the case of many partial logics a generalization of maximally consistent sets is needed to enable Henkin-style completeness proofs. Such sets are called saturated sets.

DEFINITION 22 *An sfsf $\mathfrak{Z}$ is* $\mathbf{s}$-saturated *if for every sfsf $\mathfrak{Y}$ which is $\mathbf{s}$-derivable from $\mathfrak{Z}$ has a non-empty intersection with $\mathfrak{Z}$:*

(12) $$\mathfrak{Z} \vdash_{\mathbf{s}} \mathfrak{Y} \Rightarrow \mathfrak{Z} \cap \mathfrak{Y} \neq \emptyset.$$

The collection of $\mathbf{s}$-saturated sfsf's is written as $\mathsf{Sat}_{\mathbf{s}}$.

Saturated sets are always consistent. This property is covered by taking the special case $\mathfrak{Y} = \emptyset$ in Definition 22. Maximality for the case of classical logic is induced by the classical rule $\neg$-RIGHT. This generalization of maximal consistency originates from completeness proofs for first-order intuitionistic logic with respect to an appropriate class of possible world models [1, 28].

Generalized Lindenbaum lemma The following Lemma gives an adequate generalization of the Lindenbaum lemma for classical logic.

LEMMA 23 *If $\mathbf{s}$ is a finite standard sfsf-sequent system, then for every pair $\mathfrak{X}, \mathfrak{Y}$ with $\mathfrak{X} \not\vdash_{\mathbf{s}} \mathfrak{Y}$ there exists a $\mathfrak{Z} \in \mathsf{Sat}_{\mathbf{s}}$ such that $\mathfrak{X} \subseteq \mathfrak{Z}$ and $\mathfrak{Y} \cap \mathfrak{Z} = \emptyset$.*

Proof. The construction of $\mathfrak{Z}$ follows a similar procedure as in the proof the classical Lindenbaum lemma. In fact, the construction can be performed without the use of connectives [14]. If $\{\Gamma_i\}_{i=1}^{n}$ is an enumeration of all the fsfs,[14] then define

$$\begin{aligned}
\mathfrak{Z}_0 &= \mathfrak{X} \\
\mathfrak{Z}_{n+1} &= \begin{cases} \mathfrak{Z}_n & \text{if } \mathfrak{Z}_n \not\vdash_{\mathbf{s}} \Gamma_n, \mathfrak{Y} \\ \mathfrak{Z}_n \cup \{\Gamma_n\} & \text{if } \mathfrak{Z}_n \vdash_{\mathbf{s}} \Gamma_n, \mathfrak{Y} \end{cases} \\
\mathfrak{Z} &= \bigcup_{n \in \mathbb{N}} \mathfrak{Z}_n.
\end{aligned}$$

The structural rules of $\mathbf{s}$ are needed to show that $\mathfrak{Z}$ is indeed $\mathbf{s}$-saturated, and $\mathfrak{Y} \cap \mathfrak{Z} = \emptyset$. $\lhd$

The classical Lindenbaum lemma for classical logic, i.e., every consistent set has a maximally consistent extension, can be obtained by applying $\neg$-RIGHT: $\mathfrak{X} \vdash \mathfrak{Y} \Rightarrow \mathfrak{X}, \neg\mathfrak{Y} \vdash$, and then using Lemma 23.

[14]The finiteness of fsf's, and the fact that the language is countable, imply that the set of all fsf's is also countable.

Proper saturated sets In the case of the double barreled system **Q2**-saturation turns out to be insufficient. The reason is that there exists **Q2**-saturated sfsf's which contain **Q2**-inconsistent fsf's, and it turns out to be impossible to construct models from such saturated sfsf's.

DEFINITION 24 *A* proper **s**-saturated sfsf *is an sfsf* $\mathfrak{Z} \in \mathsf{Sat}_\mathbf{s}$ *such that if* $\mathfrak{Z} \vdash_\mathbf{s} \Gamma$ *then* $\mathfrak{Z} \not\vdash_\mathbf{s} \neg\Gamma$ *for every non-empty fsf* Γ. *This collection is denoted by* $\mathsf{p\text{-}Sat}_\mathbf{s}$.

Improper **Q2**-saturated sets are easily detected. Membership of the proposition which is always undefined, $\star(= ?(p, \neg p))$, is a precise test for impropriety.

OBSERVATION 25 $\{\star\} \notin \mathfrak{Z} \Leftrightarrow \mathfrak{Z} \in \mathsf{p\text{-}Sat}_\mathbf{Q2}$ for all **Q2**-saturated sfsf's $\mathfrak{Z}$.

Proof. $\boxed{\Rightarrow:}$ If $\{\star\} \in \mathfrak{Z}$ then also $\{\neg\star\} \in \mathfrak{Z}$, since $\star \vdash_\mathbf{Q2} \neg\star$.

$\boxed{\Leftarrow:}$ Let $\Gamma \subseteq_f \mathcal{L}$ such that $\mathfrak{Z} \vdash_\mathbf{Q2} \Gamma$ and $\mathfrak{Z} \vdash_\mathbf{Q2} \neg\Gamma$. The rule AMBIG-LEFT yields $\mathfrak{Z} \vdash_\mathbf{Q2} \Gamma \cup \neg\Gamma$. Because $\Gamma \cup \neg\Gamma \vdash_\mathbf{Q2} \star$, a single application of CUT shows that it must be the case that $\{\star\} \in \mathfrak{Z}$. $\triangleleft$

Constructing models from proper saturated sets On the basis of proper **Q2**-saturated sets we define associated truth-value assignments. If $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_\mathbf{Q2}$ then the function $V_\mathfrak{Z} : \mathbb{P} \longrightarrow \{0, 1\}$ is the valuation such that

$$V_\mathfrak{Z}(p) = \begin{cases} 1 & \text{if } p \in \cup\mathfrak{Z} \\ 0 & \text{if } \neg p \in \cup\mathfrak{Z}, \end{cases} \quad \text{where } \cup\,\mathfrak{Z} \text{ is defined as } \bigcup_{\Gamma \in \mathfrak{Z}} \Gamma. \tag{13}$$

First we need to check whether $V_\mathfrak{Z}$ is a well-defined valuation.

OBSERVATION 26 $V_\mathfrak{Z} \in \mathsf{V}$.

Proof. Let $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_\mathbf{Q2}$.

Because $\mathfrak{Z} \vdash_\mathbf{Q2} \{p\}, \{\neg p\}$, we have $p \in \cup\mathfrak{Z}$ or $\neg p \in \cup\mathfrak{Z}$, for all $\mathfrak{Z} \in \mathsf{Sat}_\mathbf{Q2}$. This means that if $V_\mathfrak{Z}$ is a function, then it cannot be partial (*).

If both $p, \neg p \in \cup\mathfrak{Z}$, then we get $?(p, \neg p) \in \cup\mathfrak{Z}$ (see the next Observation 27), and so, there exists $\Gamma \in \mathfrak{Z}$ with $?(p, \neg p) \in \mathfrak{Z}$. But this means $\Gamma \vdash_\mathbf{Q2} p$ and by contraposition $\neg p \vdash_\mathbf{Q2} \neg\Gamma$, which yields $\Gamma \vdash \neg\Gamma$ by application of $\neg$-MIXED and $\neg\neg$-RIGHT. In other words, $\mathfrak{Z} \vdash \Gamma$ and $\mathfrak{Z} \vdash \neg\Gamma$, which contradicts $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_\mathbf{Q2}$. Together with (*), this result shows that $V_\mathfrak{Z}$ is a function which maps $\mathbb{P}$ to $\{0, 1\}$. $\triangleleft$

OBSERVATION 27 For every $\mathfrak{Z} \in \mathsf{Sat}_\mathbf{Q2}$ and all formulas $\varphi, \psi \in \mathcal{L}$:

$$\begin{aligned} \neg\neg\varphi \in \cup\mathfrak{Z} &\Leftrightarrow \varphi \in \cup\mathfrak{Z} \\ \varphi \wedge \psi \in \cup\mathfrak{Z} &\Leftrightarrow \varphi, \psi \in \cup\mathfrak{Z} \\ \neg(\varphi \wedge \psi) \in \cup\mathfrak{Z} &\Leftrightarrow \neg\varphi \in \cup\mathfrak{Z} \text{ or } \neg\psi \in \mathfrak{Z} \\ ?(\varphi, \psi) \in \cup\mathfrak{Z} &\Leftrightarrow \varphi, \psi \in \cup\mathfrak{Z} \\ \neg\,?(\varphi, \psi) \in \cup\mathfrak{Z} &\Leftrightarrow \neg\varphi, \neg\psi \in \cup\mathfrak{Z} \end{aligned}$$

Proof. We will only prove the ?-case. The first two are straightforward. The third and fifth can be obtained by using CONTRAP together with the second and fourth result respectively.

$\boxed{\Leftarrow:}$ If $\varphi, \psi \in \cup\mathfrak{Z}$, then there exist $\Gamma, \Delta \in \mathfrak{Z}$ such that $\varphi \in \Gamma$ and $\psi \in \Delta$. Because $\mathfrak{Z} \vdash_{\mathbf{Q2}} \Gamma$ and $\mathfrak{Z} \vdash_{\mathbf{Q2}} \Delta$, we have $\mathfrak{Z} \vdash_{\mathbf{Q2}} \Gamma \cup \Delta$ (AMBIG-RIGHT), and also $\mathfrak{Z} \vdash_{\mathbf{Q2}} \{?(\varphi, \psi)\} \cup \Gamma \setminus \{\varphi\} \cup \Delta \setminus \{\psi\}$ (?-RIGHT). Since $\mathfrak{Z} \in \mathsf{Sat}_{\mathbf{Q2}}$, the last set must be a member of $\mathfrak{Z}$, and therefore $?(\varphi, \psi) \in \cup\mathfrak{Z}$.
$\boxed{\Rightarrow:}$ Analogous through AMBIG-LEFT and ?-RIGHT. $\lhd$

The truth lemma The following lemma, the so-called truth lemma, shows that the theory of $V_{\mathfrak{Z}}$ coincides with $\mathfrak{Z}$.

LEMMA 28 *$V_{\mathfrak{Z}} \models \varphi \Leftrightarrow \varphi \in \cup\mathfrak{Z}$ and $V_{\mathfrak{Z}} \dashv \varphi \Leftrightarrow \neg\varphi \in \cup\mathfrak{Z}$ for all $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_{\mathbf{Q2}}$.*

Proof. The proof runs by induction on the construction of formulae. For propositional variables, we get the result from the definition of $V_{\mathfrak{Z}}$, while the other cases can immediately derived with application of Observation 14 on the induction hypotheses. $\lhd$

Completeness of the system Q2 Now we are ready to complete the Henkin procedure for **Q2**.

Proof of Theorem 11. Suppose that $\mathfrak{X} \nvdash_{\mathbf{Q2}} \mathfrak{Y}$. We presume that $\mathfrak{X} \neq \emptyset$ and $\mathfrak{Y} \neq \emptyset$.[15] Observation 9 yields $\overline{\mathfrak{X}} \nvdash_{\mathbf{Q2}} \mathfrak{Y}$ or $\mathfrak{X} \nvdash_{\mathbf{Q2}} \overline{\mathfrak{Y}}$.

Suppose that $\mathfrak{X} \nvdash_{\mathbf{Q2}} \overline{\mathfrak{Y}}$.

According to Lemma 23 this means that there exists a **Q2**-saturated sfsf $\mathfrak{Z}$ such that $\mathfrak{X} \subseteq \mathfrak{Z}$ (*) and $\overline{\mathfrak{Y}} \cap \mathfrak{Z} = \emptyset$ (**). Because $\mathfrak{Y} \neq \emptyset$ there exists $\Delta \in \mathfrak{Y}$, and by definition of $\overline{\mathfrak{Y}}$ also $\Delta \cup \neg\Delta \in \overline{\mathfrak{Y}}$. This means that $\star \vdash_{\mathbf{Q2}} \overline{\mathfrak{Y}}$. This implies that since $\mathfrak{Z}$ is **Q2**-saturated and $\mathfrak{Z} \cap \overline{\mathfrak{Y}} =$ that $\{\star\} \notin \mathfrak{Z}$. Observation 25 subsequently proves that $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_{\mathbf{Q2}}$.

The first conclusion (*) also validates $\cup\mathfrak{X} \subseteq \cup\mathfrak{Z}$, and therefore $V_{\mathfrak{Z}} \models \Gamma$ for all $\Gamma \in \mathfrak{X}$ (Lemma 28). Moreover, we also obtain $\Delta \nsubseteq \cup\mathfrak{Z}$ for all $\Delta \in \mathfrak{Y}$.

> In order to illustrate the validity of the last conclusion, suppose that $\Delta \subseteq \cup\mathfrak{Z}$ for certain $\Delta \in \mathfrak{Y}$. This means that for every $\delta \in \Delta$ there exists $\Delta_\delta \in \mathfrak{Z}$ with $\delta \in \Delta$. Because $\mathfrak{Z} \in \mathsf{Sat}_{\mathbf{Q2}}$, this means that $\mathfrak{Z} \vdash_{\mathbf{Q2}} \Delta_\delta$ for all $\delta \in \Delta$, hence $\mathfrak{Z} \vdash_{\mathbf{Q2}} \bigcup_{\delta \in \Delta} \Delta_\delta$ through $|\Delta|$ times application AMBIG-RIGHT, and so $\bigcup_{\delta \in \Delta} \Delta_\delta \in \mathfrak{Z}$. The fsf $\bigcup_{\delta \in \Delta} \Delta_\delta$ contains Δ and therefore must be a member of $\overline{\mathfrak{Y}}$. This conclusion contradicts (**).

Another application of the relevant truth Lemma 28 proves that $V_{\mathfrak{Z}} \not\models \Delta$ for all $\Delta \in \mathfrak{Y}$, and therefore $\mathfrak{X} \not\models_{\mathsf{V}} \mathfrak{Y}$, and in particular, $\mathfrak{X} \not\models^2_{\mathsf{V}} \mathfrak{Y}$.

Suppose that $\overline{\mathfrak{X}} \nvdash_{\mathbf{Q2}} \mathfrak{Y}$. Contraposition yields $\neg\mathfrak{Y} \nvdash_{\mathbf{Q2}} \neg\overline{\mathfrak{X}}$, and removal of double negations from the right hand sfsf in the last nonsequent settles $\neg\mathfrak{Y} \nvdash_{\mathbf{Q2}} \overline{\neg\mathfrak{X}}$. Lemma 23 shows that there exists $\mathfrak{Z} \in \mathsf{Sat}_{\mathbf{Q2}}$ such that $\neg\mathfrak{Y} \subseteq \mathfrak{Z}$ and $\overline{\neg\mathfrak{X}} \cap \mathfrak{Z} = \emptyset$. Moreover, $\{\star\} \notin \mathfrak{Z}$ since $\star \vdash_{\mathbf{Q2}} \overline{\neg\mathfrak{X}}$, and therefore, $\mathfrak{Z} \in \mathsf{p\text{-}Sat}_{\mathbf{Q2}}$. By analogy of the former case, we see that $V_{\mathfrak{Z}} \dashv \Delta$ for all $\Delta \in \mathfrak{Y}$ and $V_{\mathfrak{Z}} \not\dashv \Gamma$ for all $\Gamma \in \mathfrak{X}$, or shortly, $\mathfrak{Y} \not\dashv_{\mathsf{V}} \mathfrak{X}$. This also implies $\mathfrak{X} \not\models^2_{\mathsf{V}} \mathfrak{Y}$. $\lhd$

[15] If $\mathfrak{X} = \emptyset$, then we may freely replace $\mathfrak{X}$ by the singleton $\{\{\top\}\}$, since $\vdash_{\mathbf{Q2}} \mathfrak{Y} \Leftrightarrow \top \vdash_{\mathbf{Q2}} \mathfrak{Y}$. For the same reason, we can replace an empty $\mathfrak{Y}$ by $\{\{\bot\}\}$. The assumption of non-emptiness of the two arguments can therefore be made without loss of generality.

Appendix B: completeness of Q and its extensions

As in the completeness proof of **Q2**, we follow the Henkin procedure.

A simple construction of models from Q-saturated sets The following definition gives a first construction of canonical models induced by saturated sets. In the case of the relative ambiguity logics we do not have to exclude improper saturated sets. They do not exist.

DEFINITION 29 *Let $\mathfrak{Z}$ be a* **Q***-saturated sfsf, then*

$$V_{\mathfrak{Z}}(p) = 1 \quad \Leftrightarrow \quad \{p\} \in \mathfrak{Z} \tag{14}$$

$$\mathsf{a}_{\mathfrak{Z}}(\Gamma) \quad = \quad \begin{cases} \mathsf{Res}(\Gamma) & \text{if } \Gamma \notin \mathfrak{Z} \\ \{\varphi \in \mathsf{Res}(\Gamma) \mid \{\varphi\} \in \mathfrak{Z}\} & \text{if } \Gamma \in \mathfrak{Z}. \end{cases} \tag{15}$$

OBSERVATION 30 $\langle V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \rangle$ is a disambiguation model.

Proof. It's clear that $V_{\mathfrak{Z}}$ is a proper valuation, and furthermore, $\mathsf{a}_{\mathfrak{Z}}(\Gamma) \subseteq \mathsf{Res}(\Gamma)$. $\mathsf{a}_{\mathfrak{Z}}(\Gamma) \neq \emptyset$ if $\Gamma \neq \emptyset$ follows from AMBIG-SINGLES-LEFT. If $\mathsf{Res}(\Gamma) = \{\gamma_1, \ldots, \gamma_n\}$, then $\Gamma \vdash \{\gamma_1\}, \ldots, \{\gamma_n\}$:

$$\dfrac{\dfrac{}{\{\gamma_1\} \vdash \{\gamma_1\}}\,\text{START} \quad \ldots \quad \dfrac{}{\{\gamma_n\} \vdash \{\gamma_n\}}\,\text{START}}{\Gamma \vdash \{\gamma_1\}, \ldots, \{\gamma_n\}}\,\text{AMBIG-SINGLES-LEFT}$$

Therefore, if $\Gamma \in \mathfrak{Z}$ then $\mathfrak{Z} \vdash \{\gamma_1\}, \ldots, \{\gamma_n\}$ (by MON), and so $\{\gamma_i\} \in \mathfrak{Z}$ and $\gamma_i \in \mathsf{a}_{\mathfrak{Z}}(\Gamma)$ for certain $i \in \{1, \ldots, n\}$. If $\emptyset \neq \Gamma \notin \mathfrak{Z}$, then $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \mathsf{Res}(\Gamma) \neq \emptyset$.

The last criterion for partial disambiguations, $\mathsf{a}_{\mathfrak{Z}}(\Gamma \cup \{?(\alpha, \beta)\}) = \mathsf{a}_{\mathfrak{Z}}(\Gamma \cup \{\alpha, \beta\})$, also holds, since $\mathsf{Res}(\Gamma \cup \{?(\alpha, \beta)\}) = \mathsf{Res}(\Gamma \cup \{\alpha, \beta\})$ and $\Gamma \cdot \alpha \cdot \beta \equiv_{\mathbf{Q}} \Gamma \cdot ?(\alpha, \beta)$ (by the AMBIG-rules). The last equivalence shows that $\Gamma \cup \{\alpha, \beta\} \in \mathfrak{Z}$ iff $\Gamma \cup \{?(\alpha, \beta)\} \in \mathfrak{Z}$. $\lhd$

Truth lemma

OBSERVATION 31 $V_{\mathfrak{Z}} \models \{\varphi\} \Leftrightarrow \{\varphi\} \in \mathfrak{Z}$ for all $\varphi \in \mathcal{L}_0$.

Proof. By normal propositional logic over singletons in **Q** $\lhd$

LEMMA 32 $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \models \Gamma \Leftrightarrow \Gamma \in \mathfrak{Z}$ *for all* $\Gamma \subseteq_f \mathcal{L}$.

Proof. Let $\Gamma \notin \mathfrak{Z}$, then $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \mathsf{Res}(\Gamma)$. This implies $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \models \Gamma \Leftrightarrow V_{\mathfrak{Z}} \models \mathsf{Res}(\Gamma) \Leftrightarrow V \models \gamma$ for all $\gamma \in \mathsf{Res}(\Gamma)$. But AMBIG-SINGLES-RIGHT implies

$$\{\gamma_1\}, \ldots, \{\gamma_n\} \vdash \Gamma,$$

where $\mathsf{Res}(\Gamma) = \{\gamma_1, \ldots, \gamma_n\}$. And therefore there exists $\gamma_i \in \Gamma$ such that $\{\gamma_i\} \notin \mathfrak{Z}$, because $\Gamma \notin \mathfrak{Z}$. Observation 31 yields $V_{\mathfrak{Z}} \not\models \gamma_i$, hence, $V_{\mathfrak{Z}} \not\models \mathsf{Res}(\Gamma)$ and $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \not\models \Gamma$.

If $\Gamma \in \mathfrak{Z}$, then $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \{\gamma \in \mathsf{Res}(\Gamma) \mid \{\gamma\} \in \mathfrak{Z}\}$. Observation 31 entails $V_{\mathfrak{Z}} \models \mathsf{a}_{\mathfrak{Z}}(\Gamma)$, and so $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \models \Gamma$. $\lhd$

The completeness of Q

Proof of Theorem 16. Suppose that $\mathfrak{X} \not\vdash_{\mathbf{Q}} \mathfrak{Y}$, then there exists a **Q**-saturated sfsf $\mathfrak{Z}$ such that $\mathfrak{X} \subseteq \mathfrak{Z}$ and $\mathfrak{Y} \cap \mathfrak{Z} = \emptyset$ (Lemma 23, page 21). Lemma 32 entails $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \models \Gamma$ for all $\Gamma \in \mathfrak{Z}$ and $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \not\models \Delta$ for all $\Delta \in \mathfrak{Z}$. This means that $\mathfrak{X} \not\models_{\mathsf{D}} \mathfrak{Y}$. $\lhd$

The completeness of Ql The only thing we need to show is that **Ql**-saturated sfsf's induce left-monotonicity of the partial disambiguation in the construction in Definition 29.

LEMMA 33 *If $\mathfrak{Z}$ is a* **Ql**-*saturated sfsf, then $\mathsf{a}_{\mathfrak{Z}}$ is a left-monotonic partial disambiguation.*

Proof. Γ, Δ be a pair of fsf's, and $\mathfrak{Z}$ be a **Ql**-saturated set.

If $\Gamma \cup \Delta \notin \mathfrak{Z}$ then $\mathsf{a}_{\mathfrak{Z}}(\Gamma \cup \Delta) = \mathsf{Res}(\Gamma \cup \Delta) = \mathsf{Res}(\Gamma) \cup \mathsf{Res}(\Delta)$, and then the left-monotonicity requirement for $\mathsf{a}_{\mathfrak{Z}}$ is clearly fulfilled.

If $\Gamma \cup \Delta \in \mathfrak{Z}$, then $\mathfrak{Z} = \mathfrak{Z} \cup \{\Gamma \cup \Delta\} \vdash_{\mathbf{Ql}} \Gamma, \Delta$, because $\Gamma \cup \Delta \vdash_{\mathbf{Ql}} \Gamma, \Delta$. **Ql**-saturation of $\mathfrak{Z}$ yields $\Gamma \in \mathfrak{Z}$ or $\Delta \in \mathfrak{Z}$.

If $\Gamma \in \mathfrak{Z}$, then $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \{\varphi \in \mathsf{Res}(\Gamma) \mid \{\varphi\} \in \mathfrak{Z}\} \subseteq \{\varphi \in \mathsf{Res}(\Gamma \cup \Delta) \mid \{\varphi\} \in \mathfrak{Z}\} = \mathsf{a}_{\mathfrak{Z}}(\Gamma \cup \Delta)$. The same holds for Δ if $\Delta \in \mathfrak{Z}$. $\lhd$

Proof of Theorem 18. Suppose $\mathfrak{X} \not\vdash_{\mathbf{Ql}} \mathfrak{Y}$. Lemma 23 shows that there exists a **Ql**-saturated sfsf $\mathfrak{Z}$ such that $\mathfrak{X} \subseteq \mathfrak{Z}$ and $\mathfrak{Y} \cup \mathfrak{Z} = \emptyset$. Lemma 32 proves that $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \models \Gamma$ and $V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \not\models \Delta$ for all $\Gamma \in \mathfrak{X}, \Delta \in \mathfrak{Y}$. Lemma 33 proves that $\langle V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \rangle \in \mathsf{Dl}$, and therefore, $\mathfrak{X} \not\models_{\mathsf{Dl}} \mathfrak{Y}$. $\lhd$

As was mentioned at the end of the paper, it is not very hard to establish correspondences between additional rules for **Q**-extensions and constraints on partial disambiguations. A simple example was given in Observation 21 on page 17. The chosen logic was **Ql** with a general negation rule ¬-LEFT, while the corresponding model class consisted of models with partial disambiguations such that the disambiguation of a non-empty fsf always contradicts the disambiguation of the negation of this fsf (see Observation 21 for the formal description of this class). Below we give a proof the completeness result, which just requires a formal explanation why the 'canonical' definition of partial disambiguation as given in Definition 29 enforces the appropriate correspondence whenever we restrict ourselves to **Qll**-saturated sfsf's, where **Qll** = **Ql** + ¬-LEFT.

Proof of Observation 21. Let M be the correponding disambiguation model class. Completeness can now be proved by showing that $\langle V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \rangle \in \mathsf{M}$ for all **Qll**-saturated sets $\mathfrak{Z}$. Lemma 32 and the saturation lemma (Lemma 23) subsequently settles completeness of **Qll** with respect to M.

Let $\Gamma \neq \emptyset$ and $\Gamma \notin \mathfrak{Z}$ for a given fsf Γ and **Qll**-saturated $\mathfrak{Z}$. By definition $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \mathsf{Res}(\Gamma)$, and since $\mathsf{a}\,\mathfrak{Z}(\neg\Gamma) \neq \emptyset$, and also $\mathsf{a}\,\mathfrak{Z}(\neg\Gamma) \subseteq \mathsf{Res}(\neg\Gamma) = \neg\,\mathsf{Res}(\Gamma)$, we get $\neg\,\mathsf{a}_{\mathfrak{Z}}(\Gamma) \cap \mathsf{a}_{\mathfrak{Z}}(\neg\Gamma) \neq \emptyset$.

If $\Gamma \in \mathfrak{Z}$, then $\mathfrak{Z} \vdash_{\mathbf{Qll}} \Gamma$, and by application of ¬-LEFT also $\mathfrak{Z}, \neg\Gamma \vdash_{\mathbf{Qll}}$. Therefore, $\neg\Gamma \notin \mathfrak{Z}$, and so $\mathsf{a}_{\mathfrak{Z}}(\neg\Gamma) = \mathsf{Res}(\neg\Gamma) = \neg\,\mathsf{Res}(\Gamma)$. Analogous to the first case, this conclusion implies that $\neg\,\mathsf{a}_{\mathfrak{Z}}(\Gamma) \cap \mathsf{a}_{\mathfrak{Z}}(\neg\Gamma) \neq \emptyset$. $\lhd$

An alternative construction of models from saturated sfsf's The proofs of the completeness results for **Qr** and **Qlr** require a modification of the disambiguation models which we relate to a given saturated set to establish the relevant truth-lemmas. To see that the old construction does not work in the case that AMBIG-RIGHT rule appears, let $\Gamma = \{p\}$ and $\Delta = \{q, r\}$. Suppose that $\Gamma \cup \Delta \notin \mathfrak{Z}$, $\Delta \in \mathfrak{Z}$ and $\{q\} \notin \mathfrak{Z}$ for a given **Qr**-saturated set $\mathfrak{Z}$. First, we notice that $\Delta \not\models_{\mathsf{Dr}} q, \Gamma \cup \Delta$, so, in order to establish completeness by **Qr**-saturated sets, such a **Qr**-saturated set should exist. Clearly, $\Gamma \notin \mathfrak{Z}$ (AMBIG-RIGHT), which entails $\mathsf{a}_{\mathfrak{Z}}(\Gamma \cup \Delta) = \mathsf{Res}(\Gamma \cup \Delta)$, while $\mathsf{a}_{\mathfrak{Z}}(\Gamma) = \mathsf{Res}(\Gamma)$. If $\mathsf{a}_{\mathfrak{Z}}$ were monotonic, then also $\mathsf{a}_{\mathfrak{Z}}(\Delta) = \mathsf{Res}(\Delta)$, but on the other hand $V_{\mathfrak{Z}} \not\models \Delta$, and so $\langle V_{\mathfrak{Z}}, \mathsf{a}_{\mathfrak{Z}} \rangle \not\models \Delta$.

An advantage of AMBIG-RIGHT is that every fsf Γ has a largest subset $\Gamma_{\mathfrak{Z}}$ in a given **Qr**-saturated sfsf $\mathfrak{Z}$. This can be derived from the simple fact that $\mathfrak{Z}$ must be closed under unions:

$$\Gamma_{\mathfrak{Z}} =_{df} \bigcup_{\substack{\Gamma' \subseteq \Gamma \\ \Gamma' \in \mathfrak{Z}}} \Gamma'$$

The rule AMBIG-RIGHT implies $\Gamma_1, \Gamma_2 \vdash_{\mathbf{Qr}} \Gamma_1 \cup \Gamma_2$, and therefore, $\mathfrak{Z} \vdash_{\mathbf{Qr}} \Gamma_{\mathfrak{Z}}$ for every **Qr**-saturated sfsf $\mathfrak{Z}$ and fsf Γ.

DEFINITION 34 *Let $\mathfrak{Z}$ be a **Qr**-saturated set, the model $\langle V_{\mathfrak{Z}}, \mathsf{b}_{\mathfrak{Z}} \rangle$ is defined by:*

$$\mathsf{b}_{\mathfrak{Z}}(\Gamma) =_{df} \{\varphi \in \mathsf{Res}(\Gamma) \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) \text{ for all fsfs } \Gamma.$$

Truth lemma Legitimacy of this construction in the Henkin procedure follows from a truth lemma for the models $\langle V_{\mathfrak{Z}}, \mathsf{b}_{\mathfrak{Z}} \rangle$.

LEMMA 35 *Let $\mathfrak{Z}$ be a **Qr**-saturated set, then $\mathsf{b}_{\mathfrak{Z}}$ is a right-monotonic partial disambiguation.*

Proof. Let $\mathfrak{Z}$ be a **Qr**-saturated set. First notice that

$$\Gamma_{\mathfrak{Z}} \subseteq (\Gamma \cup \Delta)_{\mathfrak{Z}} \text{ and } \Delta_{\mathfrak{Z}} \subseteq (\Gamma \cup \Delta)_{\mathfrak{Z}}. \tag{16}$$

This is an obvious consequence of AMBIG-RIGHT and the $\mathfrak{Z}$-maximality of $(\Gamma \cup \Delta)_{\mathfrak{Z}}$ in $\Gamma \cup \Delta$. Right-monotonicity of $\mathsf{b}_{\mathfrak{Z}}$ follows from (16):

$$\begin{array}{rcl}
\mathsf{b}_{\mathfrak{Z}}(\Gamma \cup \Delta) & = & \{\varphi \in (\Gamma \cup \Delta) \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}((\Gamma \cup \Delta) \setminus (\Gamma \cup \Delta)_{\mathfrak{Z}}) = \\
& & \{\varphi \in \Gamma \mid \{\varphi\} \in \mathfrak{Z}\} \cup \{\varphi \in \Delta \mid \{\varphi\} \in \mathfrak{Z}\} \cup \\
& & \mathsf{Res}(\Gamma \setminus (\Gamma \cup \Delta)_{\mathfrak{Z}}) \cup \mathsf{Res}(\Delta \setminus (\Gamma \cup \Delta)_{\mathfrak{Z}}) \subseteq \\
& (16) & \{\varphi \in \Gamma \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) \cup \{\varphi \in \Delta \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Delta \setminus \Delta_{\mathfrak{Z}}) = \\
& & \mathsf{b}_{\mathfrak{Z}}(\Gamma) \cup \mathsf{b}_{\mathfrak{Z}}(\Delta)
\end{array}$$

$\lhd$

LEMMA 36 *Let $\mathfrak{Z}$ be a **Qr**-saturated sfsf, then $\langle V_{\mathfrak{Z}}, \mathsf{b}_{\mathfrak{Z}} \rangle \models \Gamma \Leftrightarrow \Gamma \in \mathfrak{Z}$.*

Proof. Let $\mathfrak{Z}$ be a **Qr**-saturated sfsf.

If $\Gamma \in \mathfrak{Z}$, then $\mathsf{b}_{\mathfrak{Z}}(\Gamma) = \mathsf{a}_{\mathfrak{Z}}(\Gamma)$, and then we may apply lemma 32.

If $\Gamma \notin \mathfrak{Z}$, then $\Gamma \setminus \Gamma_{\mathfrak{Z}} \neq \emptyset$, and also $\mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) \neq \emptyset$. Let $\mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) = \{\varphi_1, \ldots, \varphi_n\}$, then $\{\varphi_1\}, \ldots, \{\varphi_n\} \vdash_{\mathbf{Qr}} \Gamma \setminus \Gamma_{\mathfrak{Z}}$ (AMBIG-RIGHT[16]) Since $\Gamma \setminus \Gamma_{\mathfrak{Z}} \notin \mathfrak{Z}$, it must be the case there exists $\varphi_i \in \mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}})$ such that $\{\varphi_i\} \notin \mathfrak{Z}$. This means $V_{\mathfrak{Z}} \not\models \varphi_i$, and therefore, $V_{\mathfrak{Z}}, \mathsf{b}_{\mathfrak{Z}} \not\models \Gamma$, since $\varphi_i \in \mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) \subseteq \mathsf{b}_{\mathfrak{Z}}(\Gamma)$. $\lhd$

The completeness of Qr Together with the saturation lemma (Lemma 23), Lemmas 36 and 35 the proof of Theorem 20 analogous to the previous completeness results for **Q** and **Qr**.

Completeness of Qlr The only thing we need to show is that $\mathsf{b}_{\mathfrak{Z}}$ as defined in Definition 34 is left-monotonic if $\mathfrak{Z}$ is **Qlr**-saturated. This result, in combination with Lemma 23 and Lemma 36, proves the completeness of the system **Qlr** with respect to entailment over Dlr.

LEMMA 37 *If $\mathfrak{Z} \in \mathsf{Sat}_{\mathbf{Qlr}}$ then $\mathsf{b}_{\mathfrak{Z}}$ is left-right-monotonic.*

Proof. Right-monotonicity follows from Lemma 35. Left-monotonicity follows from:

$$(\Gamma \cup \Delta)_{\mathfrak{Z}} \vdash_{\mathbf{Qlr}} \Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}}, \Delta \cap (\Gamma \cup \Delta)_{\mathfrak{Z}}. \tag{17}$$

This is a simple side effect of AMBIG-LEFT and

$$(\Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}}) \cup (\Delta \cap (\Gamma \cup \Delta)_{\mathfrak{Z}}) = (\Gamma \cup \Delta)_{\mathfrak{Z}}.$$

Since $\mathfrak{Z}$ is a **Qlr**-saturated sfsf, we conclude $\Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}} \in \mathfrak{Z}$ or $\Delta \cap (\Gamma \cup \Delta)_{\mathfrak{Z}} \in \mathfrak{Z}$. As shown in the proof of Lemma 35, both $\Gamma_{\mathfrak{Z}} \subseteq (\Gamma \cup \Delta)_{\mathfrak{Z}}$ and $\Delta_{\mathfrak{Z}} \subseteq (\Gamma \cup \Delta)_{\mathfrak{Z}}$ hold (16). By the $\mathfrak{Z}$-maximality of $\Gamma_{\mathfrak{Z}}$ and $\Delta_{\mathfrak{Z}}$, we therefore may conclude:

$$\Gamma_{\mathfrak{Z}} = \Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}} \text{ or } \Delta_{\mathfrak{Z}} = \Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}} \tag{18}$$

Suppose that the former holds, then

$$\begin{array}{rcl}
\mathsf{b}_{\mathfrak{Z}}(\Gamma) & = & \{\varphi \in \Gamma \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Gamma \setminus \Gamma_{\mathfrak{Z}}) \subseteq \\
(18) & & \{\varphi \in \Gamma \cup \Delta \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Gamma \setminus (\Gamma \cap (\Gamma \cup \Delta)_{\mathfrak{Z}})) = \\
& & \{\varphi \in \Gamma \cup \Delta \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}(\Gamma \setminus (\Gamma \cup \Delta)_{\mathfrak{Z}}) \subseteq \\
& & \{\varphi \in \Gamma \cup \Delta \mid \{\varphi\} \in \mathfrak{Z}\} \cup \mathsf{Res}((\Gamma \cup \Delta) \setminus (\Gamma \cup \Delta)_{\mathfrak{Z}}) = \mathsf{b}_{\mathfrak{Z}}(\Gamma \cup \Delta)
\end{array}$$

The same inclusion holds for Δ if the second option in (18) were the case. $\lhd$

[16] Even AMBIG-SINGLES-RIGHT suffices here.