Semantic Theory

Week 1: Predicate Logic

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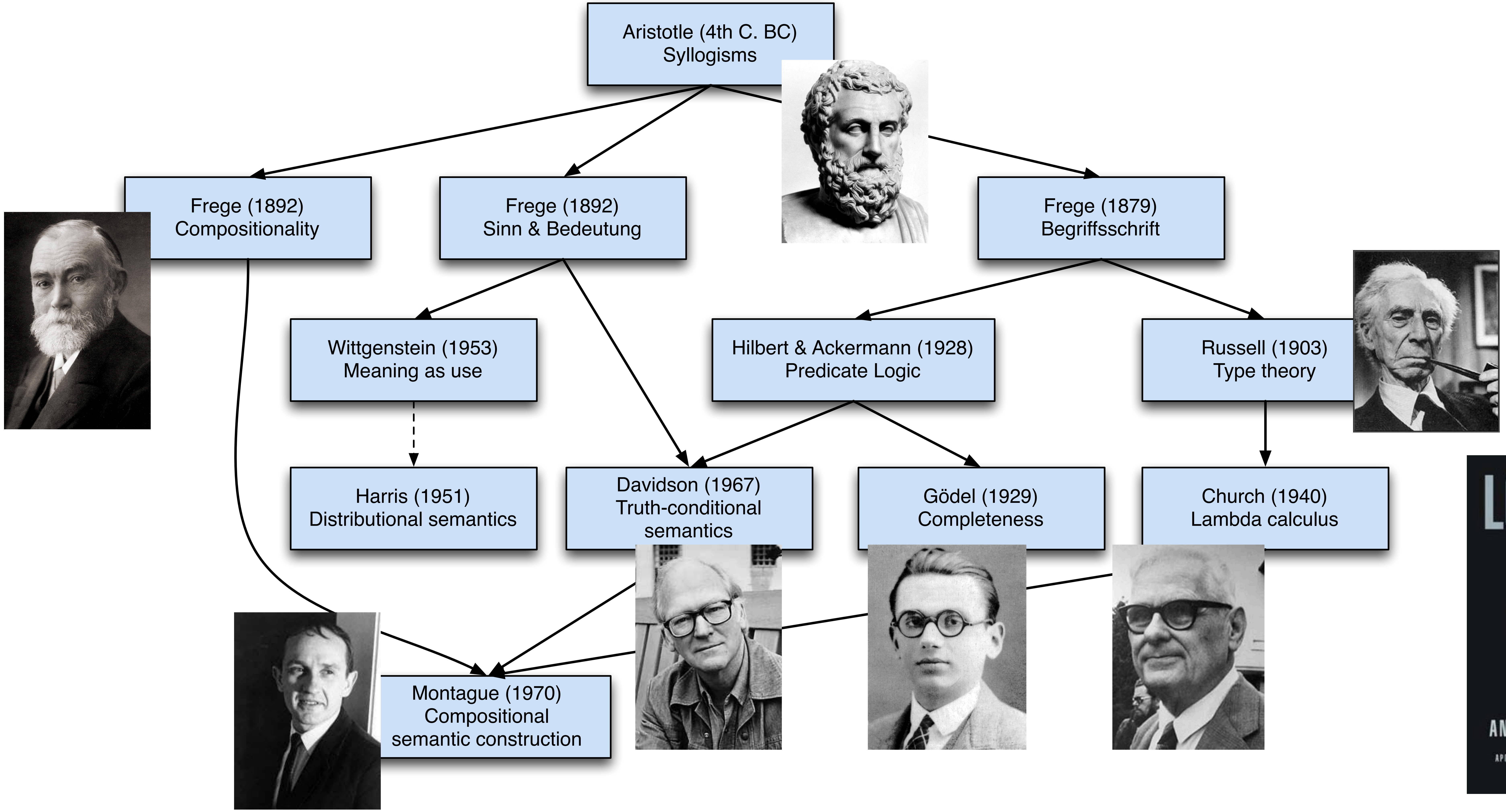
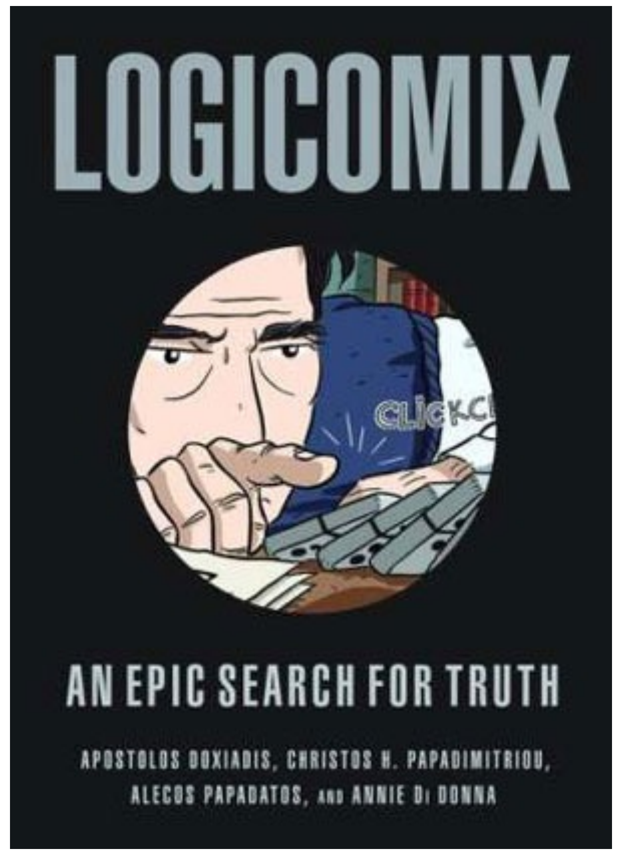
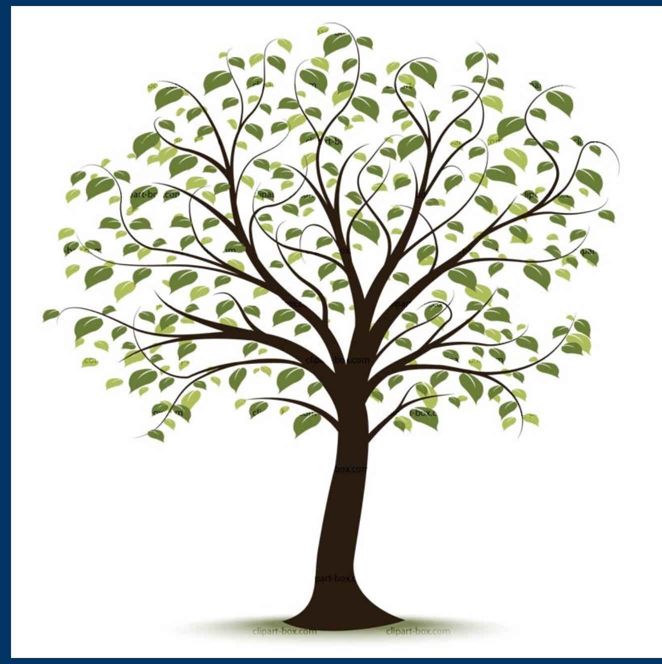
Logic & Semantic Theory

Let’s meet the players





Part I:
Sentence semantics



The most certain principle in semantics

Max Cresswell (1975): "For two sentences A and B, if in some possible situation A is true and B is false, A and B must have different meanings."

- Knowing the meaning of a (declarative) sentence requires knowing what the world would have to be like for the sentence to be true:

  Meaning = Truth Conditions

- Applied to logical representations:

  For sentence A and formula α: If there is a possible situation in which A is true and α is not, or vice versa, then α is ***not*** *an appropriate meaning representation for A.*

A central notion: Entailment

- Tina is tall and thin ⇒ Tina is tall
- Tina is tall, and Ms. Turner is not tall ⇒ Tina is not Ms.Turner
- A dog entered the room ⇒ An animal entered the room
- Tweety is a bird ⇏ Tweety can fly

Definition

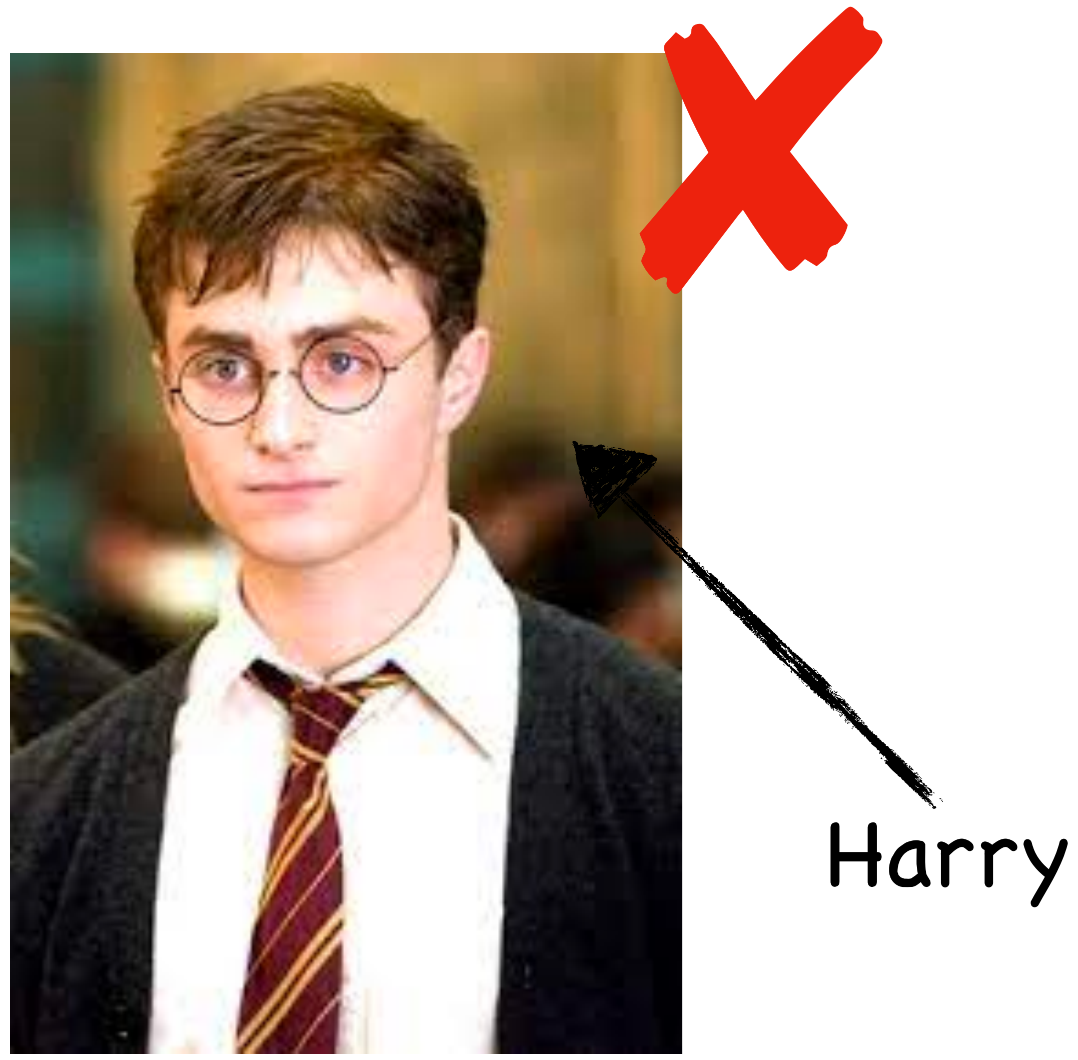
"Given an indefeasible relation between two natural language sentences S_1 and S_2, where speakers intuitively judge S_2 to be true whenever S_1 is true, we say that S_1 entails S_2, and denote it $S_1 \Rightarrow S_2$"

Truth-conditional formal semantics

- The meaning representation of a sentence must be true in exactly the same situations as the sentence itself.

"Harry is a prince" ⟷ prince(harry)

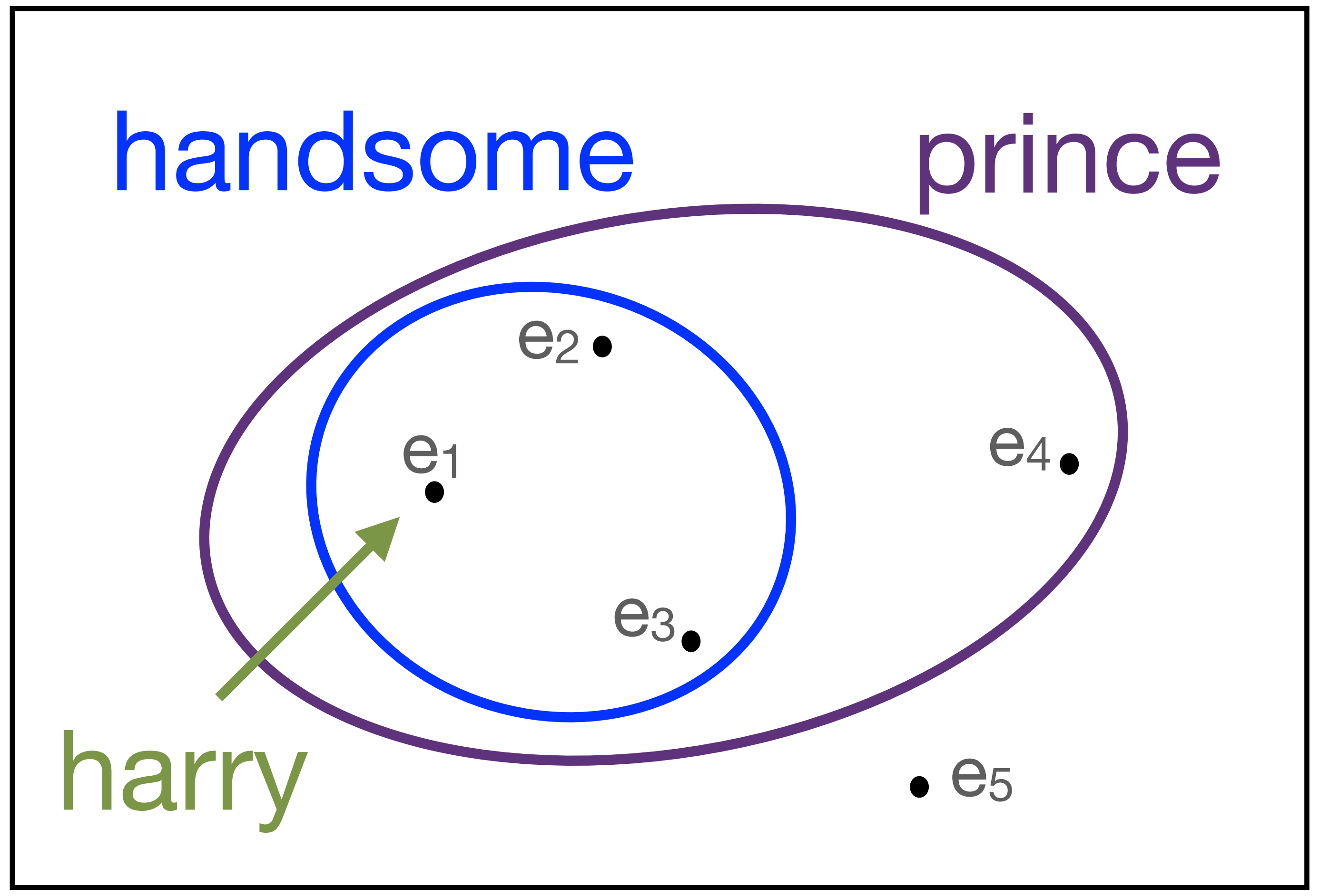
language *logic*



Model structures and formulas

- A model structure is a formal representation of a single possible situation
- A formula is a statement about model structures in a formal language
  - Formulae obtain a truth value (true / false) with respect to model structure M.

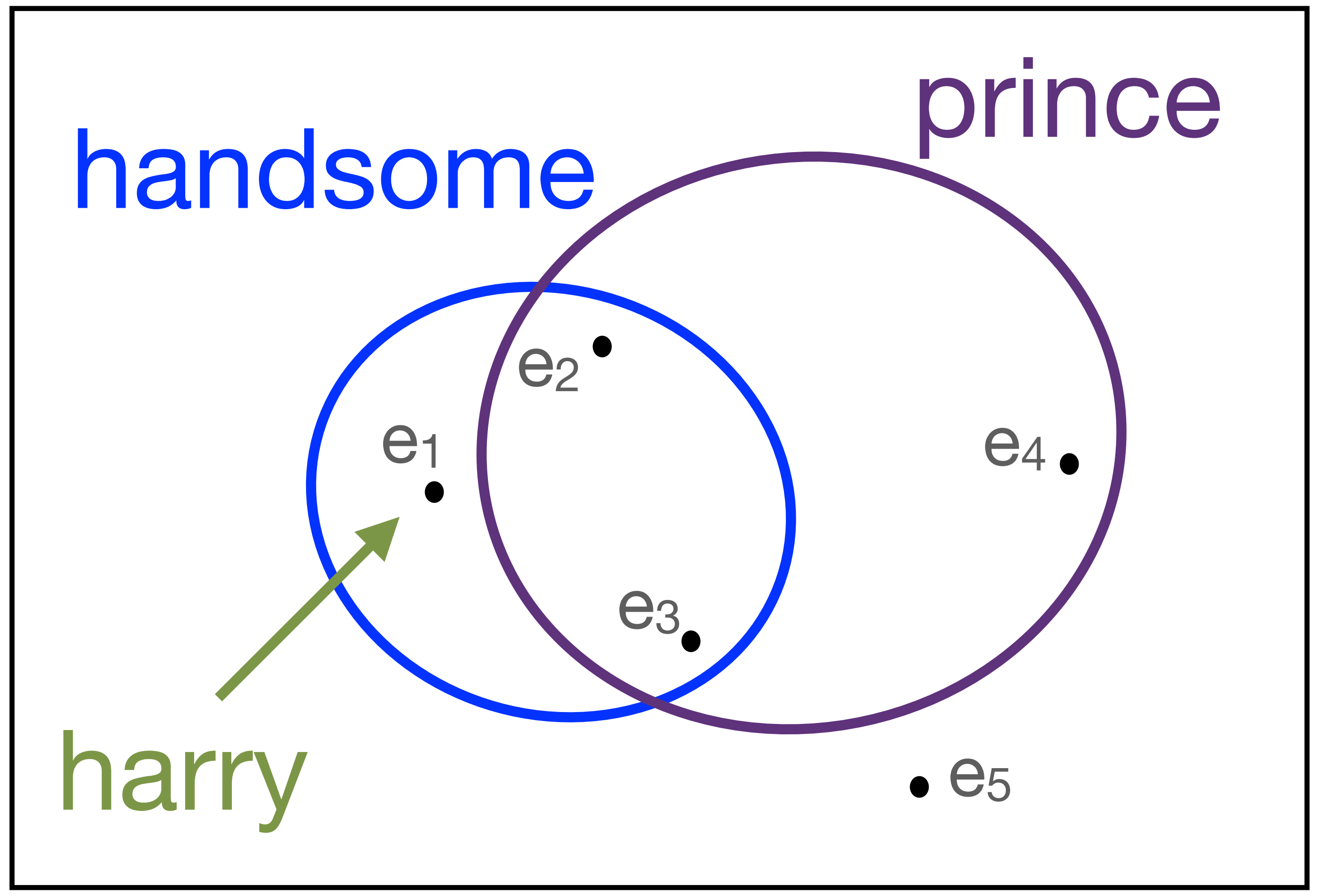
M_1:



✓ handsome(harry) ✓

✓ prince(harry) ✗

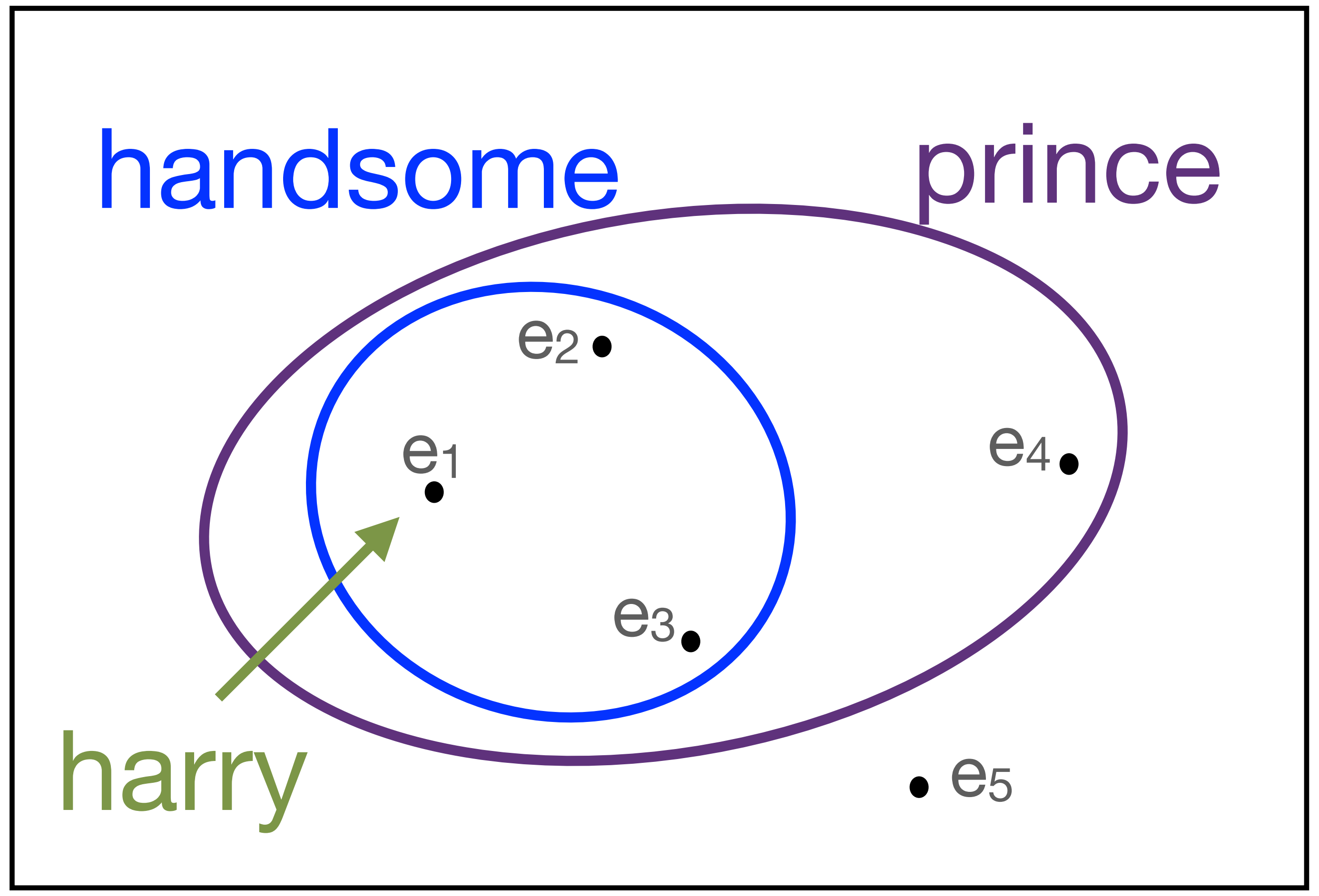
M_2:



Model structures: Definition

- Formally, a model structure M can be defined as a tuple M = ⟨U, V⟩, where:
  - U is a set of individual entities, called the *universe* (sometimes called *domain D*);
  - V is an *interpretation function* (sometimes denoted by I) that maps formula expressions onto (sets of) these entities.

M_1:



$M_1 = \langle U_1, V_1 \rangle$, where:

❑ $U_1 = \{e_1, e_2, e_3, e_4, e_5\}$

❑ $V_1(\text{handsome}) = \{e_1, e_2, e_3\}$
$V_1(\text{prince}) = \{e_1, e_2, e_3, e_4\}$
$V_1(\text{harry}) = e_1$

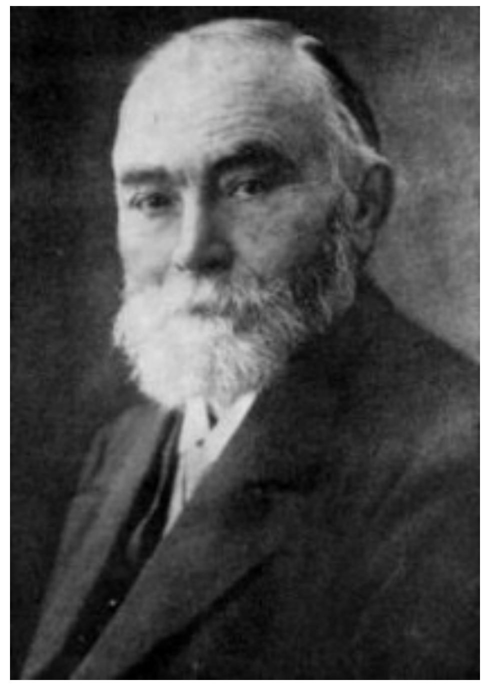
Formulas: Logical languages

- A logical language is a mathematical device that defines under what conditions a model makes a formula true.

- Propositional logic: Propositions as basic atoms
  - Syntax: propositions (p, q,..), logical connectives (¬,∧,∨,→,↔)
  - Semantics: truth tables — truth conditions, entailment
  - Limitation: propositions with internal structure

p	q	$p \& q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

- First-order predicate logic (FOL): Predicates and arguments
  - Syntax: predicates, constants and variables (*love(j,m), mortal(x), …*), quantifiers (∀,∃), logical connectives (∧, ∨, ¬, →, ↔)
  - Semantics: model structures and variable assignments



Gottlob Frege
Begriffsschrift (1879)

First-order predicate logic: Vocabulary

- **Non-logical expressions:**

  Individual constants: CON

  n-place relation constants: PRED^n, for all $n \geq 0$

- **Infinite set of individual variables:** VAR

- **Logical connectives:** $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$

- **Brackets:** (,)

First-order predicate logic: Syntax

Logic in action
Ch4: pg 26

Terms: TERM = VAR ∪ CON

Atomic formulas:

- $R(t_1,\dots, t_n)$ for $R \in PRED^n$ and $t_1, \dots, t_n \in$ TERM
- $t_1 = t_2$ for $t_1, t_2 \in$ TERM

Well-formed formula (WFF):

1. All atomic formulas are WFFs;
2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;
3. If $x \in$ VAR, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFFs;
4. Nothing else is a WFF.

FOL Formulas

Which formulas are not well-formed?

✗	1. prince	*assuming: prince ∈ PRED[1]*
✔	2. prince(x)	*free variable: x*
✗	3. prince(harry ∧ william)	*correct: prince(harry) ∧ prince(william)*
✔	4. ¬prince(harry)	*only if interpreted as: ¬(prince(harry))*
✔	5. rain → happy(kate)	*only if: rain ∈ PRED[0] (~ "it rains")*
✔	6. ∀x(rain)	*vacuous quantifier: ∀x*
✔	7. ∃x(∀x(happy(x)))	*vacuous quantifier: ∃x*
✔	8. ∀x(prince(x)) → handsome(x)	*watch the brackets! free variable: last x*

Variable binding

Logic in action
Ch4: pg 27

- Given a quantified formula $\forall x \phi$ (or $\exists x \phi$), we say that ϕ, and every part of ϕ, is in the scope of the quantifier;
- In a formula $\forall x \phi$ (or $\exists x \phi$), the quantifier occurrence binds all occurrences of x in ϕ that are not bound by any quantifier occurrence $\forall x$ or $\exists x$ *inside* ϕ;
- If a variable is not bound in formula ϕ, it occurs free in ϕ;
- A closed formula is a formula without free variables (in natural language semantics, we generally only use closed formulae);
- A quantifier $\forall x$ or $\exists x$ is called vacuous if it has no free occurrences of x in its scope.

First-order predicate logic: Semantics

Logic in action
Ch4: pp. 30-31

Interpretation of constants, predicates and variables

- FOL formulas obtain a truth value with respect to a *model structure* M and an *assignment function* g: $[\![\phi]\!]^{M,g} := [0/1]$

- First-ordered model structures are formally defined as tuples $M = \langle U_M, V_M \rangle$, where U_M is a non-empty set (the *universe*) and V_M is an interpretation function:
  - $[\![c]\!]^{M,g} = V_M(c) \in U_M$ if c is an individual constant
  - $[\![P]\!]^{M,g} = V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
  - $[\![P]\!]^{M,g} = V_M(P) \in \{0,1\}$ if P is an 0-place predicate

- The assignment function g maps variables onto elements of the universe: $g :: VAR \rightarrow U_M$
  - $[\![x]\!]^{M,g} = g(x) \in U_M$ if x is a variable

Assignment function

Mapping variables onto model entities

Logic in action
Ch4: pg. 31

An assignment function g assigns values to all variables

- g :: VAR → U_M
- We write *g[x/d]* for the assignment function *g'* that assigns *d* to *x* and assigns the same values as *g* to all other variables.

	x	**y**	**z**	**u**	**…**
g	e_1	e_2	e_3	e_4	…
g[y/e_1]	e_1	e_1	e_3	e_4	…
g[x/e_1]	e_1	e_2	e_3	e_4	…
g[y/g(z)]	e_1	e_3	e_3	e_4	…
g[y/e_1][u/e_1]	e_1	e_1	e_3	e_1	…
g[y/e_1][y/e_2]	e_1	e_2	e_3	e_4	…

Assignment function

Interpretation of variables and quantifiers

How to interpret the following sentence in model M:

- Someone is sad ↦ $\exists x(sad'(x))$

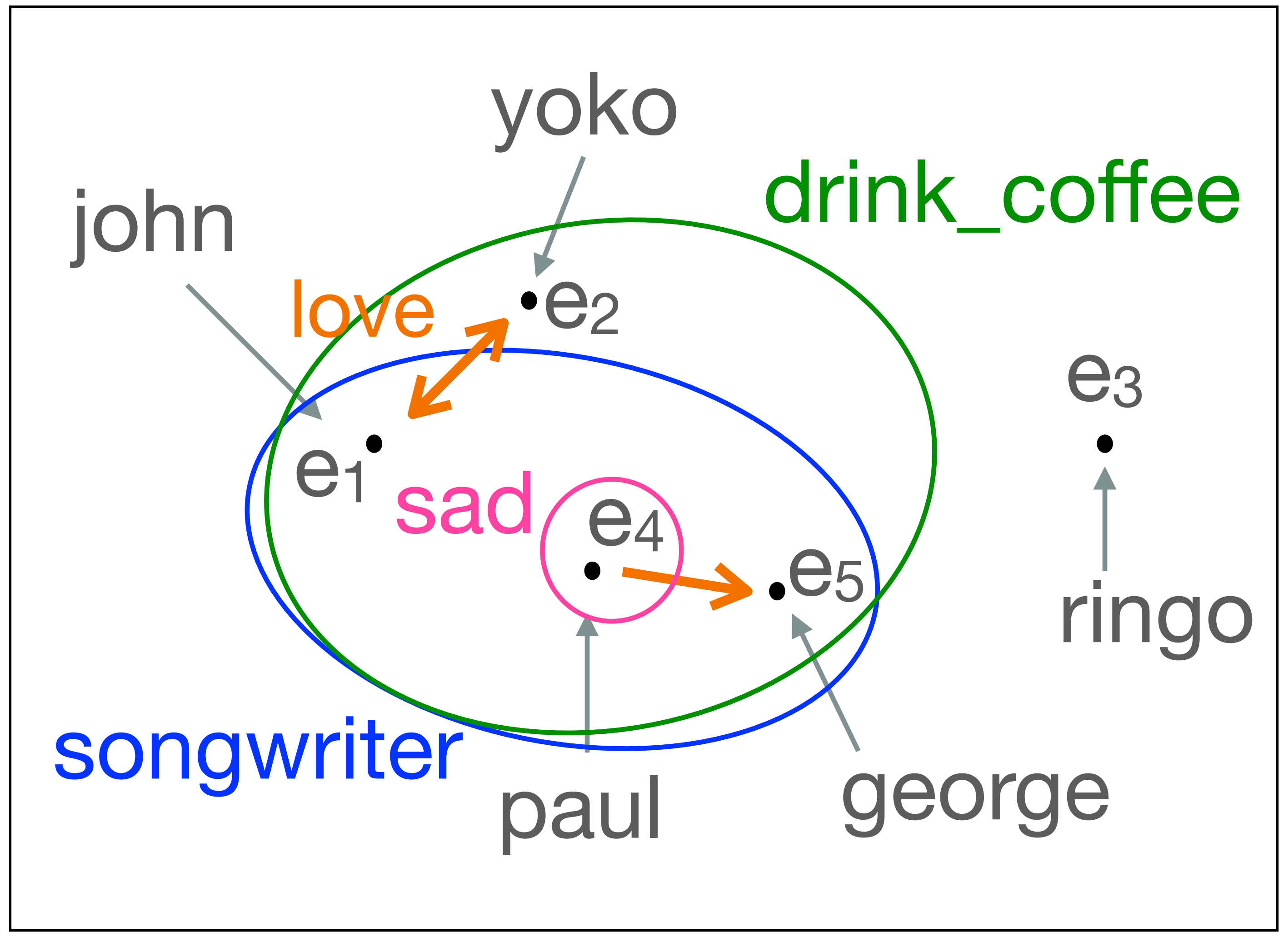
Intuition:

- find an entity in the universe for which the statement $x \in V_M(sad')$ holds: e_4
- replace x by e_4 in order to make $\exists x(sad'(x))$ true:

More formally:

- Interpret sentence relative to *assignment function* g: i.e., $[\![\exists x(sad'(x))]\!]^{M,g}$, such that $g(x) = e_4$; this can be generalised to any g' as follows: $g'[x/e_4](x) = e_4$

M_1:



$M_1 = \langle U_1, V_1 \rangle$, where:

- $U_1 = \{e_1, e_2, e_3, e_4, e_5\}$
- $V_1(john) = e_1$; $V_1(yoko) = e_2$; $V_1(ringo) = e_3$; $V_1(paul) = e_4$; $V_1(george) = e_5$
- $V_1(song\text{-}writer) = \{e_1, e_4, e_5\}$
  $V_1(drink\_coffee) = \{e_1, e_2, e_4, e_5\}$
  $V_1(love) = \{\langle e_1,e_2 \rangle, \langle e_2,e_1 \rangle, \langle e_4,e_5 \rangle\}$
  $V_1(sad) = \{e_4\}$

First-order predicate logic: Semantics

Logic in action
Ch4: pg. 34

Interpretation of formulas

Well-formed formulas are interpreted with respect to a model structure M and an assignment function g:

- $[\![R(t_1, ..., t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$
- $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
- $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$
- $[\![\exists x\phi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\phi]\!]^{M,g[x/d]} = 1$
- $[\![\forall x\phi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\phi]\!]^{M,g[x/d]} = 1$

Interpretation of formulas

Computing truth conditions and truth values

- "Every songwriter drinks coffee": $\forall x(\text{songwriter}(x) \rightarrow \text{drink\_coffee}(x))$
- Truth conditions w.r.t. M_1: $[\![\forall x(\text{songwriter}(x) \rightarrow \text{drink\_coffee}(x))]\!]^{M,g} = 1$

  iff for all *e* ∈ U: $[\![\text{songwriter}(x) \rightarrow \text{drink\_coffee}(x)]\!]^{M,g[x/e]} = 1$

  iff for all *e* ∈ U: $[\![\text{songwriter}(x)]\!]^{M,g[x/e]} = 0$ or $[\![\text{drink\_coffee}(x)]\!]^{M,g[x/e]} = 1$

  iff for all *e* ∈ U: $[\![x]\!]^{M,g[x/e]} \notin V_M(\text{songwriter})$ or $[\![x]\!]^{M,g[x/e]} \in V_M(\text{drink\_coffee})$

  iff for all *e* ∈ U: $g[x/e](x) \notin V_M(\text{songwriter})$ or $g[x/e](x) \in V_M(\text{drink\_coffee})$

  iff for all *e* ∈ U: $e \notin V_M(\text{songwriter})$ or $e \in V_M(\text{drink\_coffee})$

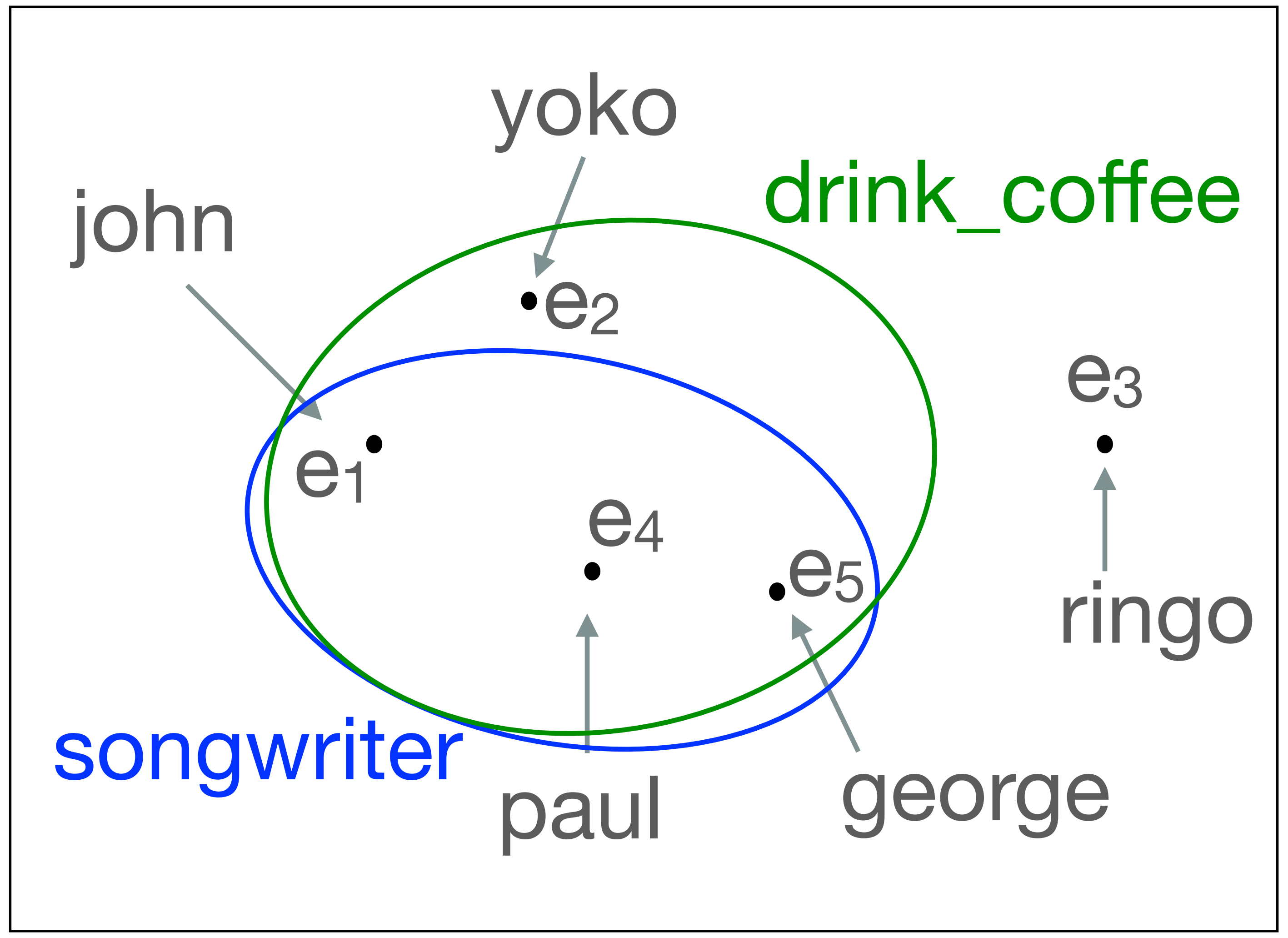
  iff $V_M(\text{songwriter}) \subseteq V_M(\text{drink\_coffee})$
- Truth value in M_1: let $\phi = \text{songwriter}(x) \rightarrow \text{drink\_coffee}(x)$

  For e = e_1, e_2, e_4, e_5,: $[\![\phi]\!]^{M1,g[x/e]} = 1$ since $e \in V_M(\text{drink\_coffee})$;

  For e = e_3: $[\![\phi]\!]^{M1,g[x/e]} = 1$ since $e \notin V_M(\text{songwriter})$.

  Therefore: $[\![\forall x(\text{songwriter}(x) \rightarrow \text{drink\_coffee}(x))]\!]^{M1,g} = 1$ for any g.

M_1:



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- $V_1(\text{song-writer}) = \{e_1, e_4, e_5\}$
  $V_1(\text{drink\_coffee}) = \{e_1, e_2, e_4, e_5\}$

Formalizing Natural Language

Exercise

1. *Bill loves Mary.*
2. *Bill reads an interesting book.*
3. *Every student reads a book.*
4. *Bill passed every exam.*
5. *Not every student answered every question.*
6. *Only Mary answered every question.*
7. *Mary is annoyed when someone is noisy.*
8. *Although nobody makes noise, Mary is annoyed.*

Try translating these sentences!

Truth, validity and entailment

- A formula φ is true in a model M iff:
  $[\![\varphi]\!]^{M,g} = 1$ for every variable assignment g
- A formula φ is valid ($\vDash \varphi$) iff:
  φ is true in all models
- A formula φ is satisfiable iff:
  there is at least one model M such that φ is true in M
- A set of formulas Γ entails formula φ ($\Gamma \vDash \varphi$) iff:
  φ is true in every model in which all formulas in Γ are true
  - the elements of Γ are called the premises or hypotheses
  - φ is called the conclusion

Logical Equivalence

Formula ϕ is logically equivalent to formula ψ ($\phi \Leftrightarrow \psi$), iff:

- $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$ for all models M and variable assignments g.

For all *closed* formulas ϕ and ψ, the following assertions are equivalent:

1. $\phi \Leftrightarrow \psi$ (logical equivalence)
2. $\phi \vDash \psi$ and $\psi \vDash \phi$ (mutual entailment)
3. $\vDash \phi \leftrightarrow \psi$ (validity of “material equivalence”)

Logical Equivalence Theorems

Propositional logic

1) $\neg\neg\phi \Leftrightarrow \phi$ — Double negation

2) $\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$ — Commutativity

3) $\phi \vee \psi \Leftrightarrow \psi \vee \phi$ — Commutativity

4) $\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ — Distributivity

5) $\phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$ — Distributivity

6) $\neg(\phi \wedge \psi) \Leftrightarrow \neg\phi \vee \neg\psi$ — de Morgan's Laws

7) $\neg(\phi \vee \psi) \Leftrightarrow \neg\phi \wedge \neg\psi$ — de Morgan's Laws

8) $\phi \rightarrow \neg\psi \Leftrightarrow \psi \rightarrow \neg\phi$ — Law of Contraposition

9) $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$

10) $\neg(\phi \rightarrow \psi) \Leftrightarrow \phi \wedge \neg\psi$

Logical Equivalence Theorems

Quantifiers

11) $\neg\forall x\phi \Leftrightarrow \exists x\neg\phi$

12) $\neg\exists x\phi \Leftrightarrow \forall x\neg\phi$

Quantifier negation

13) $\forall x(\phi \wedge \Psi) \Leftrightarrow \forall x\phi \wedge \forall x\Psi$

14) $\exists x(\phi \vee \Psi) \Leftrightarrow \exists x\phi \vee \exists x\Psi$

Quantifier distribution

15) $\forall x\forall y\phi \Leftrightarrow \forall y\forall x\phi$

16) $\exists x\exists y\phi \Leftrightarrow \exists y\exists x\phi$

Quantifier swap

17) $\exists x\forall y\phi \Rightarrow \forall y\exists x\phi$

... but not vice versa!

Logical Equivalence Theorems

Quantifiers and variables

The following equivalences are valid theorems of FOL, provided that x does not occur free in ϕ:

Here, $\phi[x/y]$ is the result of replacing all free occurrences of y in ϕ with x

18) $\exists y\phi \Leftrightarrow \exists x\phi[x/y]$
19) $\forall y\phi \Leftrightarrow \forall x\phi[x/y]$
20) $\phi \wedge \forall x\Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$
21) $\phi \wedge \exists x\Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$
22) $\phi \vee \forall x\Psi \Leftrightarrow \forall x(\phi \vee \Psi)$
23) $\phi \vee \exists x\Psi \Leftrightarrow \exists x(\phi \vee \Psi)$
24) $\phi \rightarrow \forall x\Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$
25) $\phi \rightarrow \exists x\Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$
26) $\exists x\Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$
27) $\forall x\Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$

Equivalence Transformations

Example

1. $\neg\exists x\forall y(Py \rightarrow Rxy)$ "Nobody masters every problem"
2. $\forall x\exists y(Py \wedge \neg Rxy)$ "Everybody fails to master some problem"

We show the equivalence of 1. and 2. as follows:

$\neg\exists x\forall y(Py \rightarrow Rxy)$

$\Leftrightarrow \forall x\neg\forall y(Py \rightarrow Rxy)$ (12) $\neg\exists x\phi \Leftrightarrow \forall x\neg\phi$

$\Leftrightarrow \forall x\exists y\neg(Py \rightarrow Rxy)$ (11) $\neg\forall x\phi \Leftrightarrow \exists x\neg\phi$

$\Leftrightarrow \forall x\exists y(Py \wedge \neg Rxy)$ (10) $\neg(\phi \rightarrow \psi) \Leftrightarrow \phi\wedge\neg\psi$

Reading material

- **Recommended reading:** *Logic in Action*, Chapter 4 (sections 4.5 & 4.6) — http://www.logicinaction.org
- **Further background:** Winter, *Elements of Formal Semantics*, Chapter 2 — http://www.phil.uu.nl/~yoad/efs/main.html