

Semantic Theory

Week 1 – Predicate Logic

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Part I: Sentence semantics



Formalizing sentence meaning

Goal of Semantic Theory: **formally** describe sentence meaning

- Defining differences between various linguistic **forms**
- Using **formal** mathematical methods

Truth-conditional semantics:

The (traditional) perspective on sentence meaning according to which knowing the meaning of a (declarative) sentence requires knowing what the world would have to be like for the sentence to be true:

Sentence meaning = truth-conditions

A central notion: Entailment

- Tina is tall and thin \Rightarrow Tina is tall
- Tina is tall, and Ms. Turner is not tall \Rightarrow Tina is not Ms. Turner
- A dog entered the room \Rightarrow An animal entered the room
- Tweety is a bird \nRightarrow Tweety can fly

Definition

Given an **indefeasible relation** between two natural language sentences S_1 and S_2 , where speakers intuitively judge S_2 to be true whenever S_1 is true, we say that S_1 **entails** S_2 , and denote it $S_1 \Rightarrow S_2$

From sentences to truth conditions

In traditional semantic approaches sentences are interpreted “indirectly” via a logical translation.

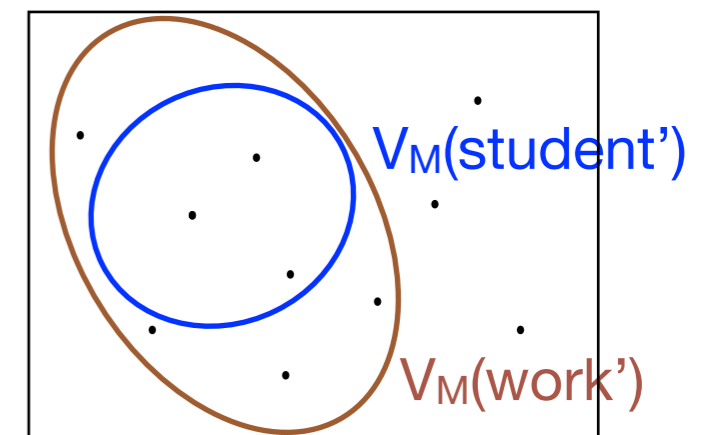
Two steps of indirect interpretation:

1. Translate sentences into logical formulas:

Every student works $\mapsto \forall x(\text{student}'(x) \rightarrow \text{work}'(x))$

2. Interpret these formulas in a logical model:

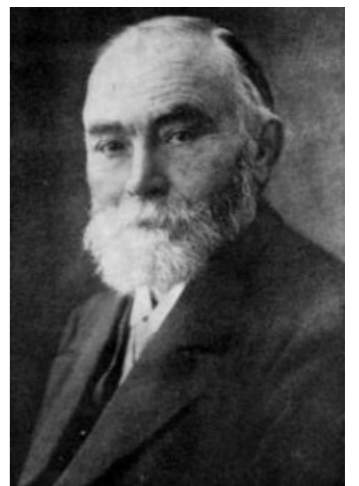
$\llbracket \forall x(\text{student}'(x) \rightarrow \text{work}'(x)) \rrbracket^{M,g} = 1$
iff $V_M(\text{student}') \subseteq V_M(\text{work}')$



Step 1: from sentence to formula

Choosing the appropriate logical formalism

- Propositional logic: Propositions as basic atoms
 - Syntax: propositions (p, q, \dots), logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$)
 - Semantics: truth tables — truth conditions, entailment
 - Limitation: propositions with internal structure
- Predicate logic: Predicates and arguments
 - Syntax: predicates & terms (love'(j',m'), mortal'(x), ...), quantifiers (\forall, \exists), logical connectives ($\wedge, \vee, \neg, \rightarrow, \leftrightarrow$)
 - Semantics: model structures and variable assignments



Gottlob Frege
Begriffsschrift (1879)

Predicate Logic: Vocabulary

Non-logical expressions:

Individual constants: CON

n-place relation constants: PRED^n , for all $n \geq 0$

Infinite set of individual variables: VAR

Logical connectives: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$

Brackets: (,)

Predicate Logic: Syntax

Terms: $TERM = VAR \cup CON$

Atomic formulas:

- $R(t_1, \dots, t_n)$ for $R \in PRED^n$ and $t_1, \dots, t_n \in TERM$
- $t_1 = t_2$ for $t_1, t_2 \in TERM$

Well-formed formula (WFF):

1. All atomic formulas are WFFs;
2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;
3. If $x \in VAR$, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFFs;
4. Nothing else is a WFF.

Variable binding

- Given a quantified formula $\forall x\phi$ (or $\exists x\phi$), we say that ϕ (and every part of ϕ) is in the **scope** of the quantifier $\forall x$ (or $\exists x$);
- A variable x is **bound** in formula ψ if x occurs in the scope of $\forall x$ or $\exists x$ in ψ ;
- If a variable is not bound in formula ψ , it occurs **free** in ψ ;
- A **closed formula** is a formula without free variables.

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Formalizing Natural Language

1. *Bill loves Mary.*

1. $\text{love}'(b',m')$

$\text{Love}'(b',m')$

2. *Bill reads an interesting book.*

$\text{read}'(b',\text{book}') \wedge \text{interesting}'(\text{book}')$

2. $\text{Ex}(\text{book}'(x) \wedge \text{interesting}'(x) \wedge \text{read}'(b',x))$

3. *Every student reads a book.*

3. $\text{Ax}(\text{student}'(x) \rightarrow \text{Ey}(\text{book}'(y) \wedge \text{read}'(x,y)))$

$\text{Ax Ey}(\text{student}'(x) \rightarrow (\text{book}'(y) \wedge \text{read}'(x,y)))$

3a. *There is a book that every student reads.*

4. *Bill passed every exam.*

4. $\text{Ax}(\text{exam}'(x) \rightarrow \text{pass}'(b',x))$

$\text{Ax}((\text{exam}'(x) \wedge \text{took}'(b',x)) \rightarrow \text{pass}'(b',x))$

5. *Not every student answered every question.*

5. $\neg \text{Ax}(\text{student}(x) \rightarrow \text{Ay}(\text{question}(y) \rightarrow \text{answer}(x,y)))$

$\neg \text{Ax Ay}((\text{student}(x) \wedge \text{question}(y)) \rightarrow \text{answer}(x,y))$

$\text{Ey}(\text{question}(y) \wedge \neg \text{Ax}(\text{student}(x) \rightarrow \text{answer}(x,y)))$

$\text{Ex Ey}(\text{student}(x) \wedge \text{question}(y) \wedge \neg \text{answer}(x,y))$

6. *Only Mary answered every question.*

6. $\text{Ax}(\text{Ay}(\text{question}(y) \rightarrow \text{answer}(x,y)) \leftrightarrow (m'=x))$

7. *Mary is annoyed when someone is noisy.*

7. $\text{Ex}(\text{noisy}(x)) \rightarrow \text{annoyed}'(m')$

$\text{Ax}(\text{noisy}(x) \rightarrow \text{annoyed}'(m'))$

8. *Although nobody makes noise, Mary is annoyed.*

8. $\neg \text{Ex}(\text{noisy}(x)) \wedge \text{annoyed}(m')$

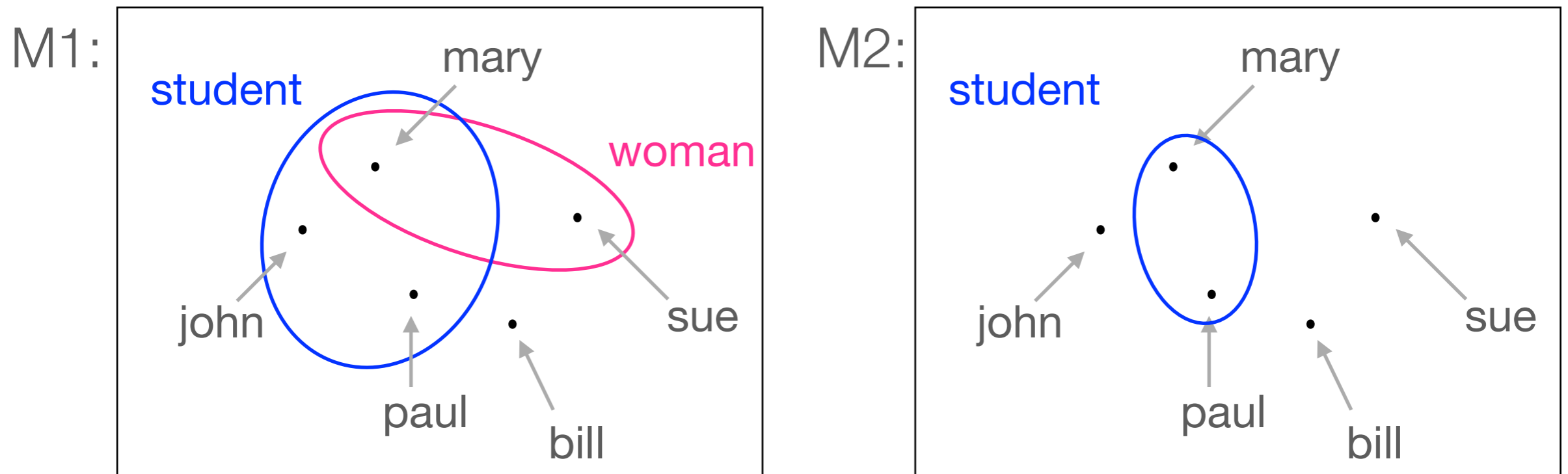
$\text{Ax}(\neg \text{noisy}(x)) \wedge \text{annoyed}(m')$

$\text{Ax}(\text{Ey}(\text{noisy}(y)) \rightarrow \text{annoyed}(x)) \wedge \neg \text{Az}(\text{noisy}(z)) \wedge \text{annoyed}(m')$

Try translating a couple of these sentences!

Step 2: Interpretation

Logical models are simplified representations of states of affairs in the world



Truth conditions are defined with respect to arbitrary logical models

John is a student : for any M , $\llbracket \text{student}'(\text{john}) \rrbracket^M = 1$ iff $V_M(\text{john}) \in V_M(\text{student}')$

$V_{M1}(\text{john}) \in V_{M1}(\text{student}')$ therefore: $\llbracket \text{student}'(\text{john}) \rrbracket^{M1} = 1$

$V_{M2}(\text{john}) \notin V_{M2}(\text{student}')$ therefore: $\llbracket \text{student}'(\text{john}) \rrbracket^{M2} = 0$

A formal description of a model

Model $M = \langle U_M, V_M \rangle$, with:

- U_M is the universe of M and
- V_M is an interpretation function

$U_M = \{e_1, e_2, e_3, e_4, e_5\}$ *universe*

$V_M(\text{john}) = e_1$

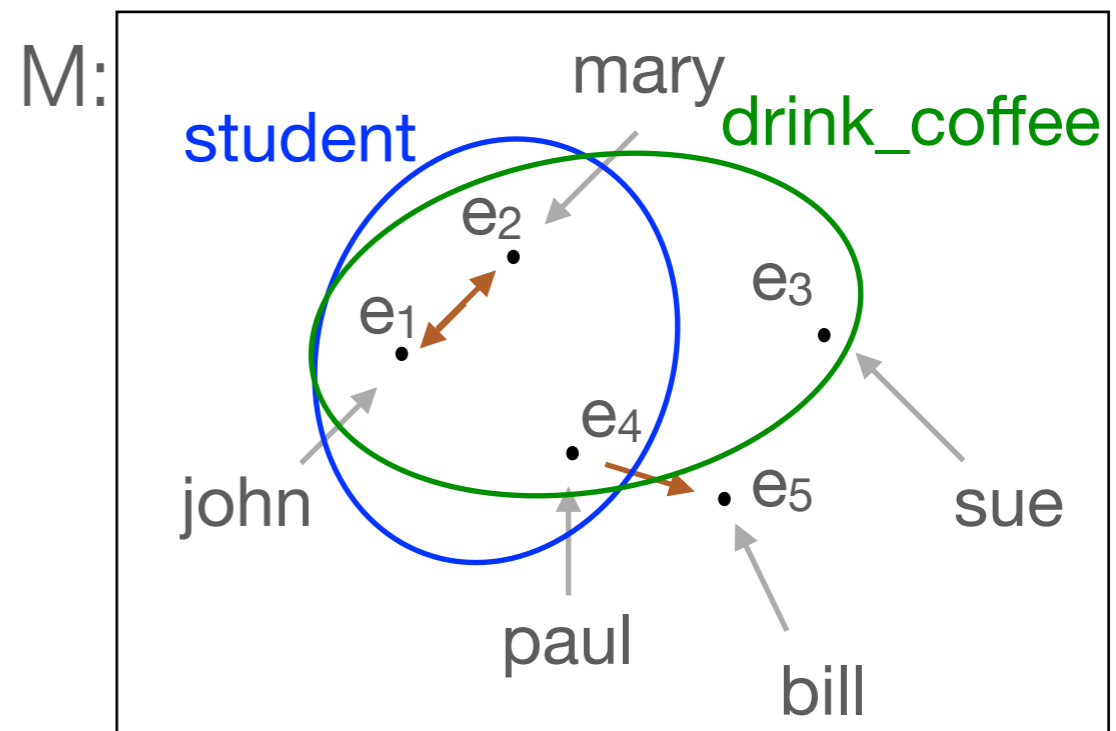
... *constants*

$V_M(\text{bill}) = e_5$

$V_M(\text{student}) = \{e_1, e_2, e_4\}$

$V_M(\text{drink_coffee}) = \{e_1, e_2, e_3, e_4\}$

$V_M(\text{love}) = \{\langle e_1, e_2 \rangle, \langle e_2, e_1 \rangle, \langle e_4, e_5 \rangle\}$



1-place predicates

2-place predicates

Interpretation in the model

V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n -ary relations over U_M to n -place predicate symbols:

- $V_M(c) \in U_M$ if c is an individual constant
- $V_M(P) \subseteq U_M^n$ if P is an n -place predicate symbol
- $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol

NB: LiA uses slightly different notation. A model is there described as the tuple $\langle \mathcal{D}, I \rangle$, where \mathcal{D} describes the Domain (here: Universe, U) and I describes the interpretation function (here: V)

Variables and quantifiers

How to interpret the following sentence in our model M:

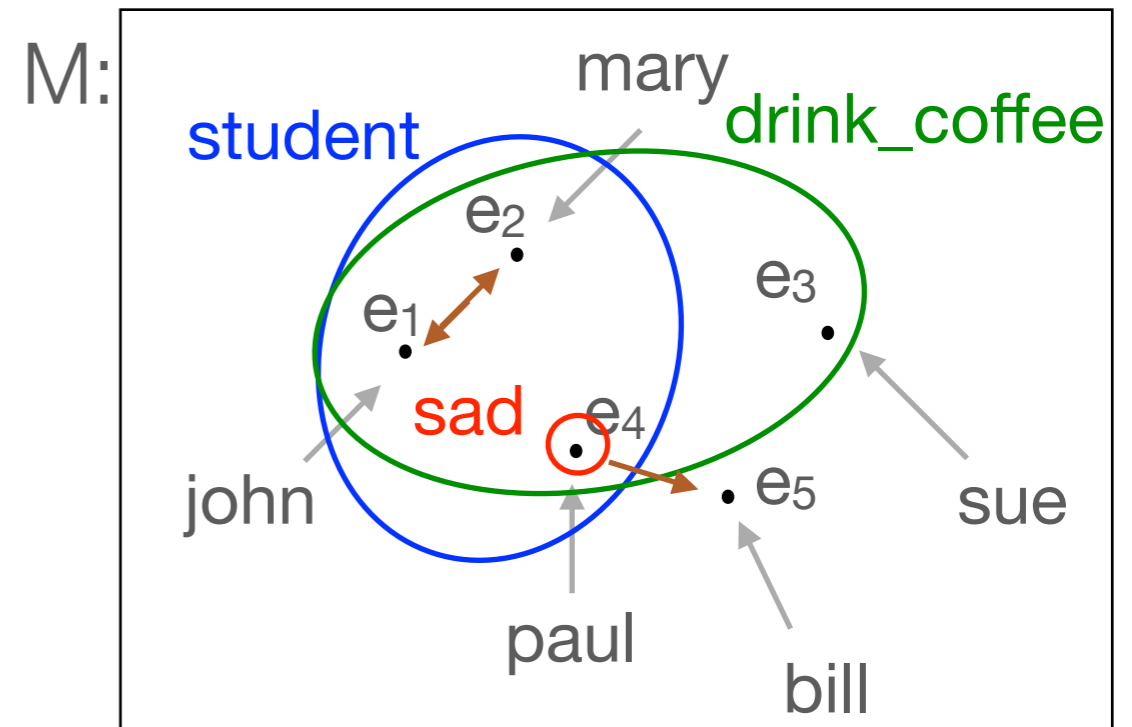
- Someone is sad $\mapsto \exists x(\text{sad}'(x))$

Intuition:

- find an entity in the universe for which the statement holds: $V_M(\text{sad}') = e_4$
- replace x by e_4 in order to make $\exists x(\text{sad}'(x))$ true

More formally:

- Interpret sentence relative to *assignment function* g : i.e., $\llbracket \exists x(\text{sad}'(x)) \rrbracket^{M,g}$, such that $g(x) = e_4$; this can be generalised to any g' as follows: $g'[x/e_4](x) = e_4$



Assignment functions

An assignment function g assigns values to all variables

- $g :: \text{VAR} \rightarrow U_M$
- We write $g[x/d]$ for the assignment function g' that assigns d to x and assigns the same values as g to all other variables.

	x	y	z	u	...
g	e_1	e_2	e_3	e_4	...
$g[y/e_1]$	e_1	e_1	e_3	e_4	...
$g[x/e_1]$	e_1	e_2	e_3	e_4	...
$g[y/g(z)]$	e_1	e_3	e_3	e_4	...
$g[y/e_1][u/e_1]$	e_1	e_1	e_3	e_1	...
$g[y/e_1][y/e_2]$	e_1	e_2	e_3	e_4	...

Interpretation of terms

Interpretation of terms with respect to a model M and a variable assignment g :

$$\llbracket \alpha \rrbracket^{M,g} = \begin{array}{ll} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{array}$$

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g :

- $\llbracket R(t_1, \dots, t_n) \rrbracket^{M,g} = 1$ iff $\langle \llbracket t_1 \rrbracket^{M,g}, \dots, \llbracket t_n \rrbracket^{M,g} \rangle \in V_M(R)$
- $\llbracket t_1 = t_2 \rrbracket^{M,g} = 1$ iff $\llbracket t_1 \rrbracket^{M,g} = \llbracket t_2 \rrbracket^{M,g}$
- $\llbracket \neg\phi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 0$
- $\llbracket \phi \wedge \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 1$ and $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \vee \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 1$ or $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \rightarrow \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 0$ or $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \leftrightarrow \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = \llbracket \psi \rrbracket^{M,g}$
- $\llbracket \exists x\phi \rrbracket^{M,g} = 1$ iff there is a $d \in U_M$ such that $\llbracket \phi \rrbracket^{M,g[x/d]} = 1$
- $\llbracket \forall x\phi \rrbracket^{M,g} = 1$ iff for all $d \in U_M$, $\llbracket \phi \rrbracket^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula ϕ is true in a model M iff:

$\llbracket \phi \rrbracket^{M,g} = 1$ for every variable assignment g

A formula ϕ is valid ($\models \phi$) iff:

ϕ is true in all models

A formula ϕ is satisfiable iff:

there is at least one model M such that ϕ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:

there is a model M such that every formula in Γ is true in M
("M satisfies Γ ," or "M is a model of Γ ")

Γ entails a formula ϕ ($\Gamma \models \phi$) iff:

ϕ is true in every model structure that satisfies Γ

Logical Equivalence

Formula ϕ is logically equivalent to formula ψ ($\phi \Leftrightarrow \psi$), iff:

- $\llbracket \phi \rrbracket^{M,g} = \llbracket \psi \rrbracket^{M,g}$ for all models M and variable assignments g .

For all *closed* formulas ϕ and ψ , the following assertions are equivalent:

1. $\phi \Leftrightarrow \psi$ (logical equivalence)
2. $\phi \models \psi$ and $\psi \models \phi$ (mutual entailment)
3. $\models \phi \leftrightarrow \psi$ (validity of “material equivalence”)

Logical Equivalence Theorems: Propositions

- 1) $\neg\neg\phi \Leftrightarrow \phi$ Double negation
- 2) $\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$ Commutativity of \wedge , \vee
- 3) $\phi \vee \psi \Leftrightarrow \psi \vee \phi$
- 4) $\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$ Distributivity of \wedge and \vee
- 5) $\phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$
- 6) $\neg(\phi \wedge \psi) \Leftrightarrow \neg\phi \vee \neg\psi$ de Morgan's Laws
- 7) $\neg(\phi \vee \psi) \Leftrightarrow \neg\phi \wedge \neg\psi$
- 8) $\phi \rightarrow \neg\psi \Leftrightarrow \psi \rightarrow \neg\phi$ Law of Contraposition
- 9) $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$
- 10) $\neg(\phi \rightarrow \psi) \Leftrightarrow \phi \wedge \neg\psi$

Logical Equivalence Theorems: Quantifiers

- | | | |
|-----|---|--------------------------|
| 11) | $\neg \forall x \phi \Leftrightarrow \exists x \neg \phi$ | Quantifier negation |
| 12) | $\neg \exists x \phi \Leftrightarrow \forall x \neg \phi$ | |
| 13) | $\forall x (\phi \wedge \psi) \Leftrightarrow \forall x \phi \wedge \forall x \psi$ | Quantifier distribution |
| 14) | $\exists x (\phi \vee \psi) \Leftrightarrow \exists x \phi \vee \exists x \psi$ | |
| 15) | $\forall x \forall y \phi \Leftrightarrow \forall y \forall x \phi$ | Quantifier Swap |
| 16) | $\exists x \exists y \phi \Leftrightarrow \exists y \exists x \phi$ | |
| 17) | $\exists x \forall y \phi \Rightarrow \forall y \exists x \phi$ | ... but not vice versa ! |

Logical Equivalence Theorems: Quantifiers (cont.)

The following equivalences are valid theorems of FOL, provided that x does not occur free in ϕ :

Here, $\phi[x/y]$ is the result of replacing all free occurrences of y in ϕ with x

$$18) \exists y\phi \Leftrightarrow \exists x\phi[x/y]$$

$$19) \forall y\phi \Leftrightarrow \forall x\phi[x/y]$$

$$20) \phi \wedge \forall x\Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$$

$$21) \phi \wedge \exists x\Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$$

$$22) \phi \vee \forall x\Psi \Leftrightarrow \forall x(\phi \vee \Psi)$$

$$23) \phi \vee \exists x\Psi \Leftrightarrow \exists x(\phi \vee \Psi)$$

$$24) \phi \rightarrow \forall x\Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$$

$$25) \phi \rightarrow \exists x\Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$$

$$26) \exists x\Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$$

$$27) \forall x\Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$$

Equivalence Transformations

(1) $\neg \exists x \forall y (Py \rightarrow Rxy)$ “Nobody masters every problem”

(2) $\forall x \exists y (Py \wedge \neg Rxy)$ “Everybody fails to master some problem”

We show the equivalence of (1) and (2) as follows:

$$\begin{aligned} \neg \exists x \forall y (Py \rightarrow Rxy) &\Leftrightarrow \forall x \neg \forall y (Py \rightarrow Rxy) && (\neg \exists x \phi \Leftrightarrow \forall x \neg \phi) \\ &\Leftrightarrow \forall x \exists y \neg (Py \rightarrow Rxy) && (\neg \forall x \phi \Leftrightarrow \exists x \neg \phi) \\ &\Leftrightarrow \forall x \exists y (Py \wedge \neg Rxy) && (\neg(\phi \rightarrow \psi) \Leftrightarrow \phi \wedge \neg \psi) \end{aligned}$$

Reading material

- **Required reading:** *Logic in Action*, Chapter 4 (sections 4.5 & 4.6) — <http://www.logicinaction.org>
- **Further background:** Winter, *Elements of Formal Semantics*, Chapter 2 — <http://www.phil.uu.nl/~yoad/efs/main.html>