Semantic Theory
Lecture 3 – Type Theory

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Summer 2015

First-order logic

First-order logic talks about:

- Individual objects
- Properties of and relations between individual objects
- Generalization across individual objects (quantification)

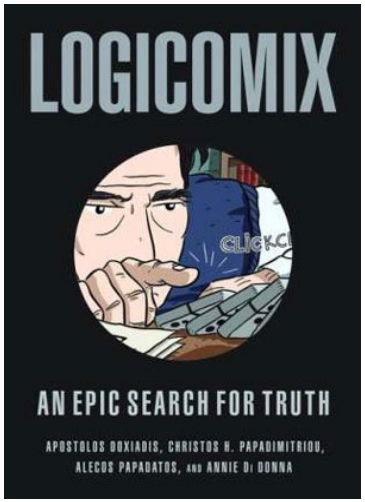
Limitations of first-order logic

FOL is not expressive enough to capture all meanings that can be expressed by basic natural language expressions:

Jumbo is a small elephant.	(Predicate modifiers)
Blond is a hair color.	(Second-order predicates)
Yesterday, it rained.	(Non-logical sentence operators)
Bill and John have the same hair color.	(Higher-order quantification)

What logical system can we use to capture this diversity?

Bertrand Russell



Q: Does the barber shave himself?

Russell's paradox

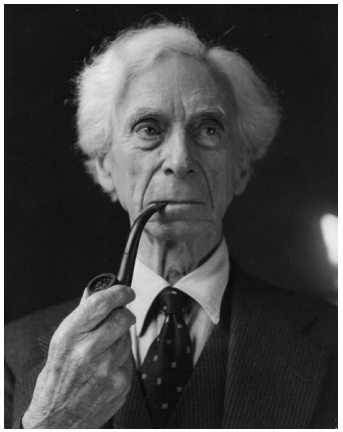
What if we extend the FOL interpretation of predicates, and interpret higher-order predicates as sets of sets of properties?

For every predicate P, we can define a set $\{x \mid P(x)\}$ containing all and only those entities for which P holds.

Then we can define a set $S = \{x \mid x \notin x\}$ representing the set of all entities that are not members of itself.

Q: does S belong to itself?

… we need a more restricted way of talking about *properties* and *relations between properties*!



Type Theory

Basic types:

- **e** – the type of individual terms (“entities”)
- **t** – the type of formulas (“truth-values”)

Complex types:

- If **σ**, **τ** are types, then ⟨**σ, τ**⟩ is a type
  (representing a functor expression that takes a **σ** type expression as argument and returns a type **τ** expression; sometimes written as: (**σ→ τ)**).

Types & Function Application

Types of first-order expressions:

- Individual constants (John, Saarbrücken) : e
- One-place predicates (sleep, walk): ⟨e, t⟩
- Two-place predicates (read, admire): ⟨e, ⟨e, t⟩⟩
- Three-place predicates (give, introduce): ⟨e, ⟨e, ⟨e, t⟩⟩⟩

Function application: Combining a functor of complex type with an appropriate argument, resulting in an expression of a less complex type: ⟨α, β⟩(α) ↦ β

- sleep'(john') :: ⟨e, t⟩(e) ⟹ t
- admire'(john') :: ⟨e,⟨e, t⟩⟩(e) ⟹ ⟨e, t⟩

More examples of types

Types of higher-order expressions:

- Predicate modifiers (expensive, poor): ⟨⟨e, t⟩, ⟨e, t⟩⟩
- Second-order predicates (hair colour): ⟨⟨e, t⟩, t⟩
- Sentence operators (yesterday, possibly, unfortunately): ⟨t, t⟩
- Degree particles (very, too): ⟨⟨⟨e, t⟩, ⟨e, t⟩⟩, ⟨⟨e, t⟩, ⟨e, t⟩⟩⟩

Tip: If σ, τ are basic types, ⟨σ, τ⟩ can be abbreviated as στ. Thus, the type of predicate modifiers and second-order predicates can be more conveniently written as ⟨et, et⟩ and ⟨et, t⟩, respectively.

Type Theory — Vocabulary

Non-logical constants:

- For every type τ a (possibly empty) set of non-logical constants CON_τ (pairwise disjoint)

Variables:

- For every type τ an infinite set of variables VAR_τ (pairwise disjoint)

Logical symbols: $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $=$

Brackets: (,)

Type Theory — Syntax

For every type τ, the set of well-formed expressions WE_τ is defined as follows:

(i) $CON_\tau \subseteq WE_\tau$ and $VAR_\tau \subseteq WE_\tau$;

(ii) If $\alpha \in WE_{\langle\sigma, \tau\rangle}$, and $\beta \in WE_\sigma$, then $\alpha(\beta) \in WE_\tau$;
(function application)

(iii) If A, B are in WE_t, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are in WE_t;

(iv) If A is in WE_t and x is a variable of arbitrary type, then $\forall xA$ and $\exists xA$ are in WE_t;

(v) If α, β are well-formed expressions of the same type, then $\alpha = \beta \in WE_t$;

(vi) Nothing else is a well-formed formula.

Function application

(ii) If $\alpha \in WE_{\langle\sigma,\tau\rangle}$, and $\beta \in WE_{\sigma}$, then $\alpha(\beta) \in WE_{\tau}$

"John is a talented piano player"

$$\dfrac{\text{john} :: e \qquad \dfrac{\text{piano\_player} :: \langle e, t\rangle \qquad \text{talented} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented(piano\_player)} :: \langle e, t\rangle}}{\text{talented(piano\_player)(john)} :: t}$$

Higher-order predicates

Higher-order quantification:

- *Bill has the same hair colour as John*

$$\exists C\,(\text{hair\_colour}(C) \wedge C(\text{bill}) \wedge C(\text{john}))$$

$\langle\langle e, t\rangle, t\rangle$ $\langle e, t\rangle$ e

Higher-order equality:

- For p, q ∈ CON_t, “p=q” expresses material equivalence: “p ↔ q”.
- For F, G ∈ $CON_{\langle e, t\rangle}$, “F=G” expresses co-extensionality: “∀x(Fx↔Gx)”
- For any formula ϕ of type *t*, ϕ=(x=x) is a representation of “ϕ is true”.

Type Theory — Semantics [1]

Let **U** be a non-empty set of entities.

The domain of possible denotations $\mathbf{D_\tau}$ for every type $\boldsymbol{\tau}$ is given by:

- $D_e = U$
- $D_t = \{0,1\}$
- $D_{\langle\sigma,\tau\rangle}$ is the set of all functions from D_σ to D_τ

Expressions of type $\boldsymbol{\tau}$ *denote elements of* $\boldsymbol{D_\tau}$

Characteristic functions

Many natural language expressions have a type ⟨**σ, t**⟩

Expressions with type ⟨**σ, t**⟩ are functions mapping elements of type **σ** to truth values: {0,1}

Such functions with a range of {0,1} are called characteristic functions, because they uniquely specify a subset of their domain $\mathbf{D_\sigma}$

The characteristic function of set M in a domain U is the function F_M: U→{0,1} such that for all a ∈ U, $F_M(a) = 1$ iff a ∈ M.

NB: For first-order predicates, the FOL representation (using sets) and the type-theoretic representation (using characteristic functions) are equivalent.

Interpretation with characteristic functions: example

For M = ⟨U, V⟩, let U consist of the persons John, Bill, Mary, Paul, and Sally. For selected types, we have the following sets of possible denotations:

- $D_t = \{0,1\}$
- $D_e = U = \{j, b, m, p, s\}$
- $D_{<e,t>} = \{ \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 1 \\ m \rightarrow 0 \\ p \rightarrow 1 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 1 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 0 \end{bmatrix}, \ldots \}$

Alternative set notation: $D_{<e,t>} = \{\{j,m,s\},\{j,b,p,s\},\{b,m\},\ldots\}$

Type Theory — Semantics [2]

A model structure for a type theoretic language is a tuple $\mathbf{M} = \langle \mathbf{U}, \mathbf{V} \rangle$ such that:

- **U** is a non-empty domain of individuals
- **V** is an interpretation function, which assigns to every $\boldsymbol{\alpha} \in \mathbf{CON}_{\tau}$ an element of $\mathbf{D}_{\tau}$ (where $\boldsymbol{\tau}$ is an arbitrary type)

The variable assignment function g assigns to every typed variable $\mathbf{v} \in \mathbf{VAR}_{\tau}$ an element of $\mathbf{D}_{\tau}$

Type Theory — Interpretation

Interpretation with respect to a model structure $M = \langle U, V \rangle$ and a variable assignment g:

- $[\![\alpha]\!]^{M,g} = V(\alpha)$ if α is a constant
  $[\![\alpha]\!]^{M,g} = g(\alpha)$ if α is a variable

- $[\![\alpha(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$

- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
  $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
  $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
  …

- $[\![\alpha = \beta]\!]^{M,g} = 1$ iff $[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}$

- $[\![\exists v\phi]\!]^{M,g} = 1$ iff there is a $d \in D_\tau$ such that $[\![\phi]\!]^{M,g[v/d]} = 1$
  $[\![\forall v\phi]\!]^{M,g} = 1$ iff for all $d \in D_\tau$: $[\![\phi]\!]^{M,g[v/d]} = 1$
  (where v is a variable of type τ)

Interpretation: Example

John is a talented piano player

$$\dfrac{\dfrac{\text{john} :: e \qquad \text{talented}(\text{piano\_player}) :: \langle e, t\rangle}{\text{talented}(\text{piano\_player})(\text{john}) :: t}}{}$$

$$\dfrac{\text{piano\_player} :: \langle e, t\rangle \qquad \text{talented} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{}$$

$[\![\text{talented}(\text{piano\_player})(\text{john})]\!]^{M,g}$

$= [\![\text{talented}(\text{piano\_player})]\!]^{M,g}([\![\text{john}]\!]^{M,g})$

$= [\![\text{talented}]\!]^{M,g}([\![\text{piano\_player}]\!]^{M,g})([\![\text{john}]\!]^{M,g})$

$= V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$

Interpretation: Example (cont.)

$[\![\textit{John is a talented piano player}]\!]^{M,g} = V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$

$V_M(\text{john}) = \text{j} \; (\in \mathbf{D_e})$

$V_M(\text{piano\_player}) = \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 1 \end{bmatrix} \; (\in \mathbf{D_{\langle e,t \rangle}}) \Leftrightarrow \{j,m,s\}$

$V_M(\text{talented}) = \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 1 \\ p \to 0 \\ s \to 1 \end{bmatrix} \to \begin{bmatrix} j \to 1 \\ b \to 0 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix} \; (\in \mathbf{D_{\langle\langle e,t\rangle\langle e,t\rangle\rangle}}) \Leftrightarrow \begin{bmatrix} \{j,m,s\} \to \{j,s\} \\ \{m,p,s\} \to \{s\} \\ \dots \end{bmatrix}$

$\begin{bmatrix} j \to 0 \\ b \to 0 \\ m \to 1 \\ p \to 1 \\ s \to 1 \end{bmatrix} \to \begin{bmatrix} j \to 0 \\ b \to 0 \\ m \to 0 \\ p \to 0 \\ s \to 1 \end{bmatrix}$

$\dots$

$V_M(\text{talented})(V_M(\text{piano\_player}))$

$V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$