Semantic Theory
Lecture 2 (Practical session) – Predicate Logic

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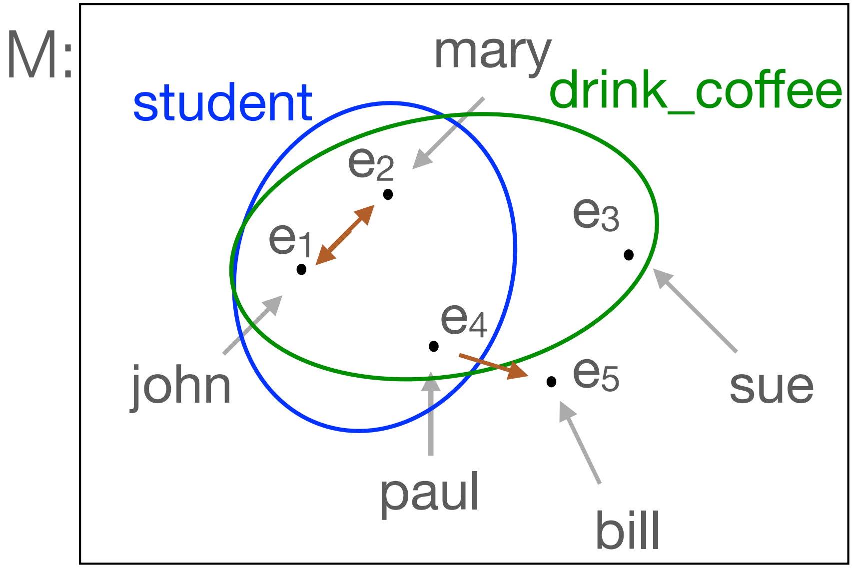
Model-theoretic interpretation

Model M = $\langle U_M, V_M \rangle$, with:

- U_M is the universe of M and
- V_M is an interpretation function



The assignment function g assigns individuals from the model to all variables

g[x/e] represents the assignment function g' in which e is assigned to x, and the values of all other variables are the same as in g.

Interpretation of terms

Interpretation of terms with respect to a model *M* and a variable assignment *g*:

$$[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$$

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g:

- $[\![R(t_1, ..., t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$
- $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
- $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$
- $[\![\exists x\phi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\phi]\!]^{M,g[x/d]} = 1$
- $[\![\forall x\phi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\phi]\!]^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula ϕ is true in a model M iff:
$[\![\phi]\!]^{M,g} = 1$ for every variable assignment g

A formula ϕ is valid ($\vDash \phi$) iff:
ϕ is true in all models

A formula ϕ is satisfiable iff:
there is at least one model M such that ϕ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:
there is a model M such that every formula in Γ is true in M
("M satisfies Γ," or "M is a model of Γ")

Γ entails a formula ϕ ($\Gamma \vDash \phi$) iff:
ϕ is true in every model structure that satisfies Γ

Determining truth and entailment

1. Find a model M, such that:

   a. $[\![\exists x(\text{rabbit'}(x) \wedge \exists y(\text{hat'}(y) \wedge \text{in'}(x, y)))]\!]^{M,g} = 1$

   b. $[\![\neg\forall x(\text{rabbit'}(x) \rightarrow \text{white'}(x))]\!]^{M,g} = 1$

2. Determine if the following entailments hold:

   a. $L(b, m) \vDash^{?} \exists x\, L(b, x)$ (with: $b, m \in CON$)

   b. $\exists x\, \forall y\, R(x,y) \vDash^{?} \forall y\, \exists x\, R(x,y)$

   c. $\forall y\, P(y) \vDash^{?} \exists y\, P(y)$

   d. $\exists x\, P(x) \wedge \exists x\, Q(x) \vDash^{?} \exists x\, (P(x) \wedge Q(x))$

Logical Equivalence

Formula ϕ is logically equivalent to formula ψ ($\phi \Leftrightarrow \psi$), iff:

- $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$ for all models M and variable assignments g.

For all *closed* formulas ϕ and ψ, the following assertions are equivalent:

1. $\phi \Leftrightarrow \psi$ (logical equivalence)
2. $\phi \vDash \psi$ and $\psi \vDash \phi$ (mutual entailment)
3. $\vDash \phi \leftrightarrow \psi$ (validity of “material equivalence”)

Logical Equivalence Theorems: Propositions

1)	$\neg\neg\phi \Leftrightarrow \phi$	Double negation
2)	$\phi\wedge\psi \Leftrightarrow \psi\wedge\phi$	Commutativity of $\wedge$, $\vee$
3)	$\phi\vee\psi \Leftrightarrow \psi\vee\phi$	
4)	$\phi\wedge(\psi\vee\chi) \Leftrightarrow (\phi\wedge\psi)\vee(\phi\wedge\chi)$	Distributivity of $\wedge$ and $\vee$
5)	$\phi\vee(\psi\wedge\chi) \Leftrightarrow (\phi\vee\psi)\wedge(\phi\vee\chi)$	
6)	$\neg(\phi\wedge\psi) \Leftrightarrow \neg\phi\vee\neg\psi$	de Morgan's Laws
7)	$\neg(\phi\vee\psi) \Leftrightarrow \neg\phi\wedge\neg\psi$	
8)	$\phi \rightarrow\neg\psi \Leftrightarrow \psi \rightarrow\neg\phi$	Law of Contraposition
9)	$\phi\rightarrow\psi \Leftrightarrow \neg\phi\vee\psi$	
10)	$\neg(\phi \rightarrow \psi) \Leftrightarrow \phi\wedge\neg\psi$	

Logical Equivalence Theorems: Quantifiers

11)	$\neg\forall x\phi \Leftrightarrow \exists x\neg\phi$	Quantifier negation
12)	$\neg\exists x\phi \Leftrightarrow \forall x\neg\phi$	
13)	$\forall x(\phi \wedge \Psi) \Leftrightarrow \forall x\phi \wedge \forall x\Psi$	Quantifier distribution
14)	$\exists x(\phi \vee \Psi) \Leftrightarrow \exists x\phi \vee \exists x\Psi$	
15)	$\forall x\forall y\phi \Leftrightarrow \forall y\forall x\phi$	Quantifier Swap
16)	$\exists x\exists y\phi \Leftrightarrow \exists y\exists x\phi$	
17)	$\exists x\forall y\phi \Rightarrow \forall y\exists x\phi$	... but not vice versa !

Logical Equivalence Theorems: Quantifiers (cont.)

The following equivalences are valid theorems of FOL, provided that x does not occur free in ϕ:

Here, ϕ[x/y] is the result of replacing all free occurrences of y in ϕ with x

18) $\exists y\phi \Leftrightarrow \exists x\phi[x/y]$

19) $\forall y\phi \Leftrightarrow \exists x\phi[x/y]$

20) $\phi \wedge \forall x\Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$

21) $\phi \wedge \exists x\Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$

22) $\phi \vee \forall x\Psi \Leftrightarrow \forall x(\phi \vee \Psi)$

23) $\phi \vee \exists x\Psi \Leftrightarrow \exists x(\phi \vee \Psi)$

24) $\phi \rightarrow \forall x\Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$

25) $\phi \rightarrow \exists x\Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$

26) $\exists x\Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$

27) $\forall x\Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$

Equivalence Transformations

(1) $\neg\exists x\forall y(Py \to Rxy)$ "Nobody masters every problem"

(2) $\forall x\exists y(Py \wedge \neg Rxy)$ "Everybody fails to master some problem"

We show the equivalence of (1) and (2) as follows:

$\neg\exists x\forall y(Py \to Rxy) \Leftrightarrow \forall x\neg\forall y(Py \to Rxy)$ $(\neg\exists x\phi \Leftrightarrow \forall x\neg\phi)$

$\Leftrightarrow \forall x\exists y\neg(Py \to Rxy)$ $(\neg\forall x\phi \Leftrightarrow \exists x\neg\phi)$

$\Leftrightarrow \forall x\exists y(Py \wedge \neg Rxy)$ $(\neg(\phi \to \psi) \Leftrightarrow \phi\wedge\neg\psi)$

Background reading material

- Gamut: Logic, Language, and Meaning Vol I/II — Chapter 2
- For a more basic introduction, see:
  http://www.logicinaction.org — Chapter 4