Semantic Theory
Lecture 2 – Predicate Logic

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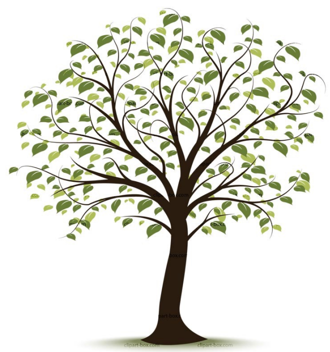
Part I:
Sentence semantics



Sentence meaning

Truth-conditional semantics:

to know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true:

Sentence meaning = truth-conditions

Indirect interpretation:

1. Translate sentences into logical formulas:
   Every student works $\mapsto \forall x(\text{student'}(x) \rightarrow \text{work'}(x))$

2. Interpret these formulas in a logical model:
   $[\![\forall x(\text{student'}(x) \rightarrow \text{work'}(x))]\!]^{M,g} = 1$ iff $V_M(\text{student'}) \subseteq V_M(\text{work'})$

Step 1: Translation

Limits of propositional logic: propositions with internal structure

Every man is mortal.

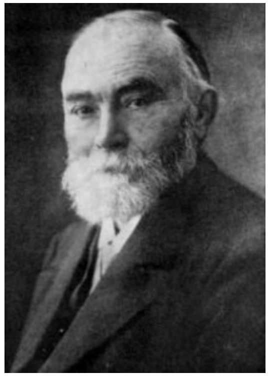
Socrates is a man.

Therefore, Socrates is mortal.

Solution: first-order predicate logic

predicates are expressions that contain *arguments* (that can be quantified over)

predication & quantification over *individuals*



Gottlob Frege

Predicate Logic: Vocabulary

Non-logical expressions:

Individual constants: CON

n-place relation constants: $PRED^n$, for all $n \geq 0$

Infinite set of individual variables: VAR

Logical connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$

Brackets: (,)

Predicate Logic: Syntax

Terms: TERM = VAR ∪ CON

Atomic formulas:

- $R(t_1, \ldots, t_n)$ for $R \in PRED^n$ and $t_1, \ldots, t_n \in$ TERM
- $t_1 = t_2$ for $t_1, t_2 \in$ TERM

Well-formed formula (WFF):

1. All atomic formulas are WFFs;
2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;
3. If $x \in$ VAR, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFFs;
4. Nothing else is a WFF.

Variable binding

- Given a quantified formula $\forall x \phi$ (or $\exists x \phi$), we say that ϕ (and every part of ϕ) is in the **scope** of the quantifier $\forall x$ (or $\exists x$);
- A variable x is **bound** in formula ψ if x occurs in the scope of $\forall x$ or $\exists x$ in ψ;
- If a variable is not bound in formula ψ, it occurs **free** in ψ;
- A **closed formula** is a formula without free variables.

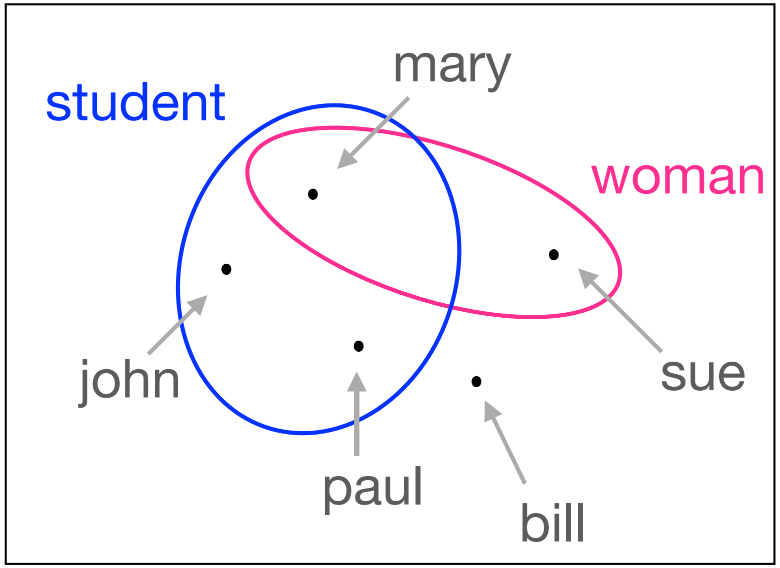
Formalizing Natural Language

1. *Bill loves Mary.*
2. *Bill reads a book.*
3. *Bill reads an interesting book.*
4. *Every student reads a book.*
5. *Bill passed every exam.*
6. *Not every student passed the exam.*
7. *Not every student answered every question.*
8. *Only Bill answered every question.*
9. *Mary is annoyed if someone is noisy.*
10. *Although nobody makes noise, Mary is annoyed.*

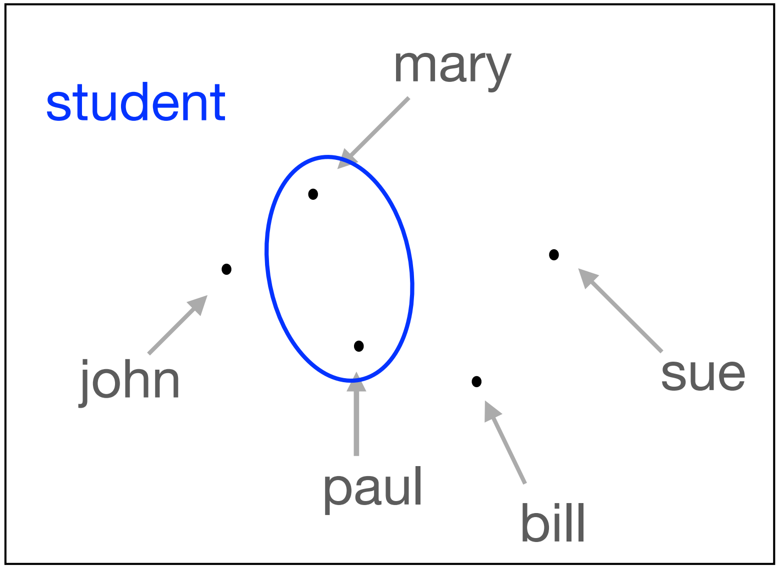
Step 2: Interpretation

Logical models are simplified representations of the state of affairs in the world

M1:



M2:



John is a student : $[\![\text{student'(john)}]\!]^M = 1$ *iff* $V_M(\text{john}) \in V_M(\text{student'})$

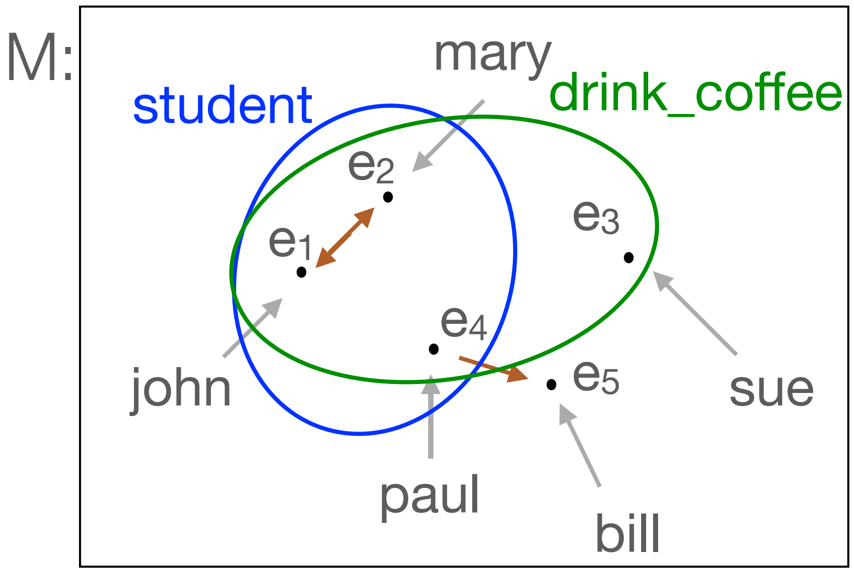
$V_{M1}(\text{john}) \in V_{M1}(\text{student'})$ therefore: $[\![\text{student'(john)}]\!]^M = 1$

$V_{M2}(\text{john}) \notin V_{M2}(\text{student'})$ therefore: $[\![\text{student'(john)}]\!]^M = 0$

A formal description of a model

Model M = ⟨U_M, V_M⟩, with:

- U_M is the universe of M and
- V_M is an interpretation function



U_M = {e1, e2, e3, e4, e5} **universe**

V_M(john) = e1
…
V_M(bill) = e5

constants

V_M(student) = {e1, e2, e4}
V_M(drink_coffee) = {e1, e2, e3, e4}

1-place predicates

V_M(love) = {⟨e1,e2⟩, ⟨e2,e1⟩, ⟨e4,e5⟩}

2-place predicates

Interpretation in the model

V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:

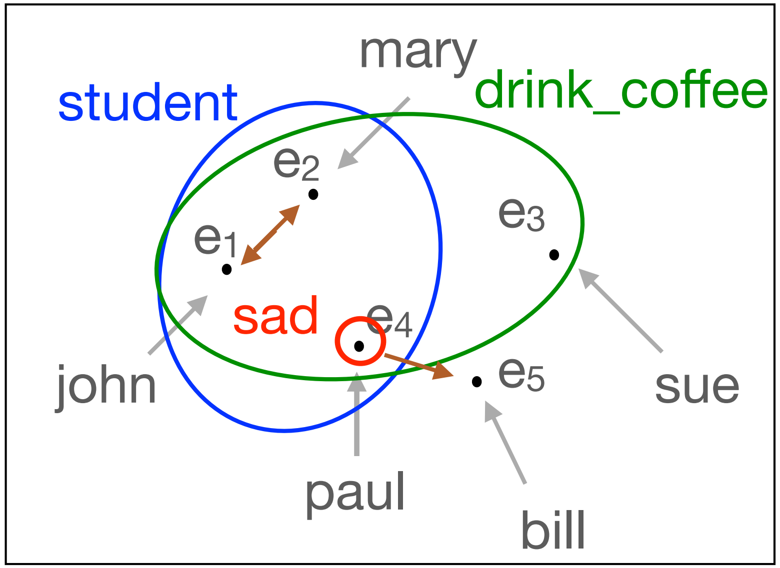
- $V_M(c) \in U_M$ if c is an individual constant
- $V_M(P) \subseteq U_M{}^n$ if P is an n-place predicate symbol
- $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol

Variables and quantifiers

How to interpret the following sentence in our model M:

- Someone is sad ↦ ∃x(sad'(x))

M:



Intuition:

- find an entity in the universe for which the statement holds: $V_M(sad') = e_4$

- replace x in by e_4 in order to make ∃x(sad'(x)) true

More formally:

- Interpret sentence relative to *assignment function* g: $[\![\exists x(sad'(x))]\!]^{M,g}$ such that $g(x) = e_4$ or $g[x/e_4]$

Assignment functions

An assignment function g assigns values to all variables

- g :: VAR → U_M
- We write *g[x/d]* for the assignment function *g'* that assigns *d* to *x* and assigns the same values as *g* to all other variables.

	x	y	z	u	…
g	e_1	e_2	e_3	e_4	…
g[y/e_1]	e_1	e_1	e_3	e_4	…
g[x/e_1]	e_1	e_2	e_3	e_4	…
g[y/g(z)]	e_1	e_3	e_3	e_4	…
g[y/e_1][u/e_1]	e_1	e_1	e_3	e_1	…
g[y/e_1][y/e_2]	e_1	e_2	e_3	e_4	…

Interpretation of terms

Interpretation of terms with respect to a model *M* and a variable assignment *g*:

$[\![\alpha]\!]^{M,g} =$	$V_M(\alpha)$	if α is an individual constant
	$g(\alpha)$	if α is a variable

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g:

- $[\![R(t_1, ..., t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$
- $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
- $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$
- $[\![\exists x\phi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\phi]\!]^{M,g[x/d]} = 1$
- $[\![\forall x\phi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\phi]\!]^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula ϕ is true in a model M iff:
  $[\![\phi]\!]^{M,g} = 1$ for every variable assignment g

A formula ϕ is valid ($\vDash \phi$) iff:
  ϕ is true in all models

A formula ϕ is satisfiable iff:
  there is at least one model M such that ϕ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:
  there is a model M such that every formula in Γ is true in M
  ("M satisfies Γ," or "M is a model of Γ")

Γ entails a formula ϕ ($\Gamma \vDash \phi$) iff:
  ϕ is true in every model structure that satisfies Γ

Background reading material

- Gamut: Logic, Language, and Meaning Vol I/II — Chapter 2
- For a more basic introduction, see:
  http://www.logicinaction.org — Chapter 4