Semantic Theory
Lecture 2 – Formal Foundations

Manfred Pinkal & Stefan Thater
FR 4.7 Allgemeine Linguistik (Computerlinguistik)
Universität des Saarlandes

Summer 2012

Today

- First-order Predicate Logic
  - Syntax
  - Semantics
- Formalizing natural language

Sentence Meaning (recap)

- **Truth-conditional semantics:** to know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true.
- **Sentence meaning = truth-conditions**
  - $[\![\text{Every student works}]\!]^{M,g} = 1$ iff. every student works
- **Indirect interpretation** by translating the sentence into some logical formula
  - Every student works $\mapsto \forall x(\text{student}'(x) \rightarrow \text{work}'(x))$

Predicate Logic – Vocabulary

- **Non-logical expressions:**
  - Individual constants: CON
  - n-place relation constants: $PRED^n$, for all $n \geq 0$
- **Infinite set of individual variables:** VAR
- **Logical connectives:** $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$
- **Brackets:** (,)

Predicate Logic – Syntax

- **Terms:** TERM = VAR $\cup$ CON
- **Atomic formulas:**
  - $R(t_1,\ldots, t_n)$ for $R \in PRED^n$ and $t_1, \ldots, t_n \in$ TERM
  - $t_1 = t_2$ for $t_1, t_2 \in$ TERM
- **Well-formed formulas:** the smallest set WFF such that
  - all atomic formulas are WFF
  - if φ and ψ are WFF, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are WFF
  - if $x \in$ VAR, and φ is a WFF, then $\forall x\varphi$ and $\exists x\varphi$ are WFF

Formalizing natural language

(1) *Bill loves Mary*

(2) *Bill reads a book*

(3) *Bill passed every exam*

(4) *Every student passed [the exam]*

(5) *Every student reads a book*

(6) *Bill and Mary are friends*

Free and Bound Variables

- If $\forall x\varphi$ ($\exists x\varphi$) is a subformula of a formula ψ, then φ is the **scope** of this occurrence of $\forall x$ ($\exists x$) in ψ.
- An *occurrence* of variable x in a formula φ is **free in φ** if this occurrence of x does not fall within the scope of a quantifier $\forall x$ or $\exists x$ in φ.
- If $\forall x\psi$ ($\exists x\psi$) is a subformula of φ and x is free in ψ, then this occurrence of x is **bound by** this occurrence of the quantifier $\forall x$ ($\exists x$).
- A **sentence** is a formula without free variables.

Predicate Logic – Semantics

- Expressions of Predicate Logic are interpreted relative to **model structures** and **variable assignments**.
- Model structures are our “mathematical picture” of the world: They provide interpretations for the non-logical symbols (predicate symbols, individual constants).
- Variable assignments provide interpretations for variables.

Model structures

- **Model structure:** $M = \langle U_M, V_M \rangle$
  - U_M is non-empty set – the “universe”
  - V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:
    - $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
    - $V_M(c) \in U_M$ if c is an individual constant
- **Assignment function** for variables $g: VAR \rightarrow U_M$

Model structures – Example

$M = \langle U_M, V_M \rangle$

$U_M = \{ r_1, r_2, h_1, h_2 \}$

$V_M(\text{vincent}) = r_1$

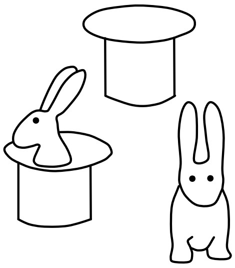
$V_M(\text{mia}) = r_2$

$V_M(\text{rabbit}) = \{ r_1, r_2 \}$

$V_M(\text{white}) = \{ r_2 \}$

$V_M(\text{hat}) = \{ h_1, h_2 \}$

$V_M(\text{in}) = \{ \langle r_1, h_1 \rangle \}$



Model structures – Example

$M = \langle U_M, V_M \rangle$

$U_M = \{ r_1, r_2, h_1, h_2 \}$

$V_M(\text{vincent}) = r_1$

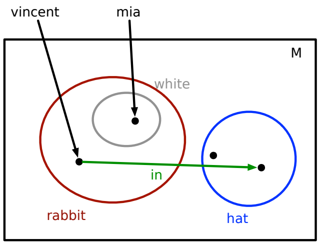
$V_M(\text{mia}) = r_2$

$V_M(\text{rabbit}) = \{ r_1, r_2 \}$

$V_M(\text{white}) = \{ r_2 \}$

$V_M(\text{hat}) = \{ h_1, h_2 \}$

$V_M(\text{in}) = \{ \langle r_1, h_1 \rangle \}$



Interpretation (Terms)

- **Interpretation of terms** with respect to a model structure M and a variable assignment g:
  - $[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$

Interpretation (Formulas)

- **Interpretation of formulas** with respect to a model structure M and variable assignment g:

$⟦R(t_1, …, t_n)⟧^{M,g} = 1$ iff $⟨⟦t_1⟧^{M,g}, …, ⟦t_n⟧^{M,g}⟩ \in V_M(R)$

$⟦t_1 = t_2⟧^{M,g} = 1$ iff $⟦t_1⟧^{M,g} = ⟦t_2⟧^{M,g}$

$⟦\neg\varphi⟧^{M,g} = 1$ iff $⟦\varphi⟧^{M,g} = 0$

$⟦\varphi \wedge \psi⟧^{M,g} = 1$ iff $⟦\varphi⟧^{M,g} = 1$ and $⟦\psi⟧^{M,g} = 1$

$⟦\varphi \vee \psi⟧^{M,g} = 1$ iff $⟦\varphi⟧^{M,g} = 1$ or $⟦\psi⟧^{M,g} = 1$

$⟦\varphi \rightarrow \psi⟧^{M,g} = 1$ iff $⟦\varphi⟧^{M,g} = 0$ or $⟦\psi⟧^{M,g} = 1$

$⟦\varphi \leftrightarrow \psi⟧^{M,g} = 1$ iff $⟦\varphi⟧^{M,g} = ⟦\psi⟧^{M,g}$

$⟦\exists x\varphi⟧^{M,g} = 1$ iff there is a $d \in U_M$ such that $⟦\varphi⟧^{M,g[x/d]} = 1$

$⟦\forall x\varphi⟧^{M,g} = 1$ iff for all $d \in U_M$, $⟦\varphi⟧^{M,g[x/d]} = 1$

Variable assignments

- We write **g[x/d]** for the assignment that assigns d to x and assigns the same values as g to all other variables.
  - g[x/d](y) = d, if x = y
  - g[x/d](y) = g(y), if x ≠ y

	x	**y**	**z**	**u**	**…**
g	a	b	c	d	…
g[x/a]	a	b	c	d	…
g[y/a]	a	a	c	d	…
g[y/g(z)]	a	c	c	d	…
g[y/a][u/a]	a	a	c	a	…
g[y/a][y/b]	a	b	c	d	…

A rabbit is in a hat

- $⟦\exists x(\text{rabbit}'(x) \wedge \exists y(\text{hat}'(y) \wedge \text{in}'(x, y)))⟧^{M,g} = 1$
  - iff … [⇒ whiteboard]

No rabbit is white

- $[\![\neg\exists x(\text{rabbit'}(x) \land \text{white'}(x))]\!]^{M,g} = 1$
  - iff … [⇒ whiteboard]

Truth &al.

- **A formula φ is true in a model structure M** iff $[\![\varphi]\!]^{M,g} = 1$ for every variable assignment g
- **A formula φ is valid (⊨ φ)** iff φ is true in all model structures
- **A formula φ is satisfiable** iff there is at least one model structure M such that φ is true in M
- **A set of formulas Γ is (simultaneously) satisfiable** iff there is a model structure M such that every formula in Γ is true in M ("M satisfies Γ," or "M is a model of Γ")
- **Γ entails a formula φ (Γ ⊨ φ)** iff φ is true in every model structure that satisfies Γ

Entailment?

(1) $L(b, m) \models^{?} \exists x\, L(b, x)$ (b, m ∈ CON)

(2) $\exists x \forall y\, R(x,y) \models^{?} \forall y \exists x\, R(x,y)$

(3) $\forall y\, P(y) \models^{?} \exists y\, P(y)$

(4) $\exists x\, P(x) \land \exists x\, Q(x) \models^{?} \exists x\, (P(x) \land Q(x))$

Formalizing natural language

(1) *Bill reads a book*

(2) *Bill reads an interesting book*

(3) *Not every student answered every question*

(4) *Only Bill answered every question*

(5) *Two students flunked*

(6) *Mary is annoyed if someone is noisy*

(7) *Although nobody makes noise, Mary is annoyed*

Literature

- L.T.F. Gamut (1991): *Logic, Language and Meaning, Vol I.* University of Chicago Press. Chapter 3