

Semantic Theory

Lecture 2: Formal Foundations

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Summer 2011

Today

- **First-order Predicate Logic**
 - Syntax
 - Semantics
- **Formalizing natural language**

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Sentence Meaning

- **Truth-conditional semantics:** to know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true.
- **Sentence meaning = truth-conditions**
 - $\llbracket \text{Every student works} \rrbracket^{M,g} = 1$ iff. every student works
- **Indirect interpretation** by translating the sentence into some logical formula
 - Every student works $\mapsto \forall x(\text{student}'(x) \rightarrow \text{work}'(x))$

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Predicate Logic - Vocabulary

- **Non-logical expressions:**
 - Individual constants: CON
 - n-place relation constants: PRED^n , for all $n \geq 0$
- **Infinite set of individual variables:** VAR
- **Logical connectives:** $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$
- **Brackets:** (,)

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Predicate Logic - Syntax

- **Terms:** $\text{TERM} = \text{VAR} \cup \text{CON}$
- **Atomic formulas:**
 - $R(t_1, \dots, t_n)$ for $R \in \text{PRED}^n$ and $t_1, \dots, t_n \in \text{TERM}$
 - $t_1 = t_2$ for $t_1, t_2 \in \text{TERM}$
- **Well-formed formulas:** the smallest set WFF such that
 - all atomic formulas are WFF
 - if ϕ and ψ are WFF, then $\neg\phi, (\phi \wedge \psi), (\phi \vee \psi), (\phi \rightarrow \psi), (\phi \leftrightarrow \psi)$ are WFF
 - if $x \in \text{VAR}$, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFF

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Free and Bound Variables

- If $\forall x\phi$ ($\exists x\phi$) is a subformula of a formula ψ , then ϕ is the **scope** of this occurrence of $\forall x$ ($\exists x$) in ψ .
- An *occurrence* of variable x in a formula ϕ is **free in ϕ** if this occurrence of x does not fall within the scope of a quantifier $\forall x$ or $\exists x$ in ϕ .
- If $\forall x\psi$ ($\exists x\psi$) is a subformula of ϕ and x is free in ψ , then this occurrence of x is **bound by** this occurrence of the quantifier $\forall x$ ($\exists x$).
- A **sentence** is a formula without free variables.

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Predicate Logic - Semantics

- Expressions of Predicate Logic are interpreted relative to **model structures** and **variable assignments**.
- Model structures are our “mathematical picture” of the world. They provide interpretations for the non-logical symbols (predicate symbols, individual constants).
- Variable assignments provide interpretations for variables.

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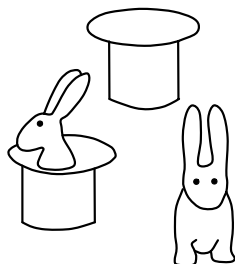
Model structures

- **Model structure:** $M = \langle U_M, V_M \rangle$
 - U_M is non-empty set - the “universe”
 - V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:
 - $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
 - $V_M(c) \in U_M$ if c is an individual constant
- **Assignment function** for variables $g: \text{VAR} \rightarrow U_M$

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Model structures - Example

$M = \langle U_M, V_M \rangle$
 $U_M = \{ r_1, r_2, h_1, h_2 \}$
 $V_M(\text{vincent}) = r_1$
 $V_M(\text{mia}) = r_2$
 $V_M(\text{rabbit}) = \{ r_1, r_2 \}$
 $V_M(\text{white}) = \{ r_2 \}$
 $V_M(\text{hat}) = \{ h_1, h_2 \}$
 $V_M(\text{in}) = \{ \langle r_1, h_1 \rangle \}$



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Model structures - Example

$$M = (U_M, V_M)$$

$$U_M = \{ r_1, r_2, h_1, h_2 \}$$

$$V_M(\text{vincent}) = r_1$$

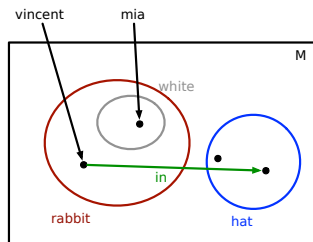
$$V_M(\text{mia}) = r_2$$

$$V_M(\text{rabbit}) = \{ r_1, r_2 \}$$

$$V_M(\text{white}) = \{ r_2 \}$$

$$V_M(\text{hat}) = \{ h_1, h_2 \}$$

$$V_M(\text{in}) = \{ \langle r_1, h_1 \rangle \}$$



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Interpretation (Terms)

- **Interpretation of terms** with respect to a model structure M and a variable assignment g :
 - $\llbracket \alpha \rrbracket^{M,g} = V_M(\alpha)$ if α is an individual constant
 - $\llbracket \alpha \rrbracket^{M,g} = g(\alpha)$ if α is a variable

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Interpretation (Formulas)

- **Interpretation of formulas** with respect to a model structure M and variable assignment g :

$$\llbracket R(t_1, \dots, t_n) \rrbracket^{M,g} = 1 \text{ iff } \langle \llbracket t_1 \rrbracket^{M,g}, \dots, \llbracket t_n \rrbracket^{M,g} \rangle \in V_M(R)$$

$$\llbracket t_1 = t_2 \rrbracket^{M,g} = 1 \text{ iff } \llbracket t_1 \rrbracket^{M,g} = \llbracket t_2 \rrbracket^{M,g}$$

$$\llbracket \neg \varphi \rrbracket^{M,g} = 1 \text{ iff } \llbracket \varphi \rrbracket^{M,g} = 0$$

$$\llbracket \varphi \wedge \psi \rrbracket^{M,g} = 1 \text{ iff } \llbracket \varphi \rrbracket^{M,g} = 1 \text{ and } \llbracket \psi \rrbracket^{M,g} = 1$$

$$\llbracket \varphi \vee \psi \rrbracket^{M,g} = 1 \text{ iff } \llbracket \varphi \rrbracket^{M,g} = 1 \text{ or } \llbracket \psi \rrbracket^{M,g} = 1$$

$$\llbracket \varphi \rightarrow \psi \rrbracket^{M,g} = 1 \text{ iff } \llbracket \varphi \rrbracket^{M,g} = 0 \text{ or } \llbracket \psi \rrbracket^{M,g} = 1$$

$$\llbracket \varphi \leftrightarrow \psi \rrbracket^{M,g} = 1 \text{ iff } \llbracket \varphi \rrbracket^{M,g} = \llbracket \psi \rrbracket^{M,g}$$

$$\llbracket \exists x \varphi \rrbracket^{M,g} = 1 \text{ iff there is a } d \in U_M \text{ such that } \llbracket \varphi \rrbracket^{M,g[x/d]} = 1$$

$$\llbracket \forall x \varphi \rrbracket^{M,g} = 1 \text{ iff for all } d \in U_M, \llbracket \varphi \rrbracket^{M,g[x/d]} = 1$$

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Variable assignments

- We write $g[x/d]$ for the assignment that assigns d to x and assigns the same values as g to all other variables.
 - $g[x/d](y) = d$, if $x = y$
 - $g[x/d](y) = g(y)$, if $x \neq y$

	x	y	z	u	...
g	a	b	c	d	...
$g[x/a]$	a	b	c	d	...
$g[y/a]$	a	a	c	d	...
$g[y/g(z)]$	a	c	c	d	...
$g[y/a][u/a]$	a	a	c	a	...
$g[y/a][y/b]$	a	b	c	d	...

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A rabbit is in a hat

- $\llbracket \exists x(\text{rabbit}'(x) \wedge \exists y(\text{hat}'(y) \wedge \text{in}'(x, y))) \rrbracket^{M,g} = 1$
 - iff ... [\Rightarrow whiteboard]

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No rabbit is white

- $\llbracket \neg \exists x(\text{rabbit}'(x) \wedge \text{white}'(x)) \rrbracket^{M,g} = 1$
 - iff ... [\Rightarrow whiteboard]

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Truth & al.

- **A formula φ is true in a model structure M** iff $\llbracket \varphi \rrbracket^{M,g} = 1$ for every variable assignment g
- **A formula φ is valid ($\models \varphi$)** iff φ is true in all model structures
- **A formula φ is satisfiable** iff there is at least one model structure M such that φ is true in M
- **A set of formulas Γ is (simultaneously) satisfiable** iff there is a model structure M such that every formula in Γ is true in M (“ M satisfies Γ ,” or “ M is a model of Γ ”)
- **Γ entails a formula φ ($\Gamma \models \varphi$)** iff φ is true in every model structure that satisfies Γ

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Entailment?

- (1) $L(b, m) \models? \exists x L(b, x)$ ($b, m \in \text{CON}$)
- (2) $\exists x \forall y R(x, y) \models? \forall y \exists x R(x, y)$
- (3) $\forall y P(y) \models? \exists y P(y)$
- (4) $\exists x P(x) \wedge \exists x Q(x) \models? \exists x (P(x) \wedge Q(x))$

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Let's talk about students

- (1) *Bill is a student*
- (2) *Bill reads an interesting book*
- (3) *Every student passed [the exam]*
- (4) *Only Bill answered every question*
- (5) *Two students flunked*
- (6) *Mary is annoyed if someone is noisy*
- (7) *Although nobody makes noise, Mary is annoyed*

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Literature

- L.T.F. Gamut (1991): *Logic, Language and Meaning, Vol I.* University of Chicago Press. Chapter 3