Semantic Theory
Intensional Logic

Manfred Pinkal
Stefan Thater

Universität des Saarlandes

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Today ...

- The principle of extensionality
- Intensional propositional logic
- Intensional predicate logic

Substitutions and Extensionality

- Substitutions: If φ is a subformula of χ, we write $[\psi/\varphi]\chi$ for the formula obtained from χ by replacing φ in χ by ψ.
- The principle of extensionality: If φ and ψ are equivalent sentences, then χ and $[\psi/\varphi]\chi$ are equivalent
  - $\Phi \leftrightarrow \Psi \models \chi \leftrightarrow [\Phi/\Psi]\chi$

Substitutability?

(1) *Barack Obama is married to Michelle Obama.*

(2) *Barack Obama is the American president.*

(3) *The American president is married to Michelle Obama.*

Substitutability?

(1) *Barack Obama has always been married to Michelle Obama.*

(2) *Barack Obama is the American president.*

(3) *The American president has always been married to Michelle Obama.*

Substitutability?

(1) *In 1963 the president of the United States was assassinated in Dallas, Texas.*

(2) *Barack Obama is the president of the United States.*

(3) *In 1963 Barack Obama was assassinated in Dallas, Texas.*

Substitutability?

(1) *By constitution, the American president is the Supreme Commander of the Armed Forces.*

(2) *Barack Obama is the American president.*

(3) *By constitution, Barack Obama is the Supreme Commander of the Armed Forces.*

Substitutability?

(1) *The detective knows that the thief entered through the skylight.*

(2) *Biggles is the thief.*

(3) *The detective knows that Biggles entered through the skylight.*

say, discover, believe, know, suspect, …

Substitutability?

(1) *Nine necessarily exceeds seven.*

(2) *Nine is the number of planets.*

(3) *The number of planets necessarily exceeds seven.*

Intensional Logic

- Intensional logic is an extension of "standard" logic (propositional logic and predicate logic)
- Come with "richer" model structures
  - Modal logic – possible worlds
  - Temporal logic – points in time
  - […]

Modal Propositional Logic

- Extend propositional logic with two modal operators:
  - □A – *it is necessary that A, necessarily A*
  - ◇A – *it is possible that A, possibly A*

Syntax

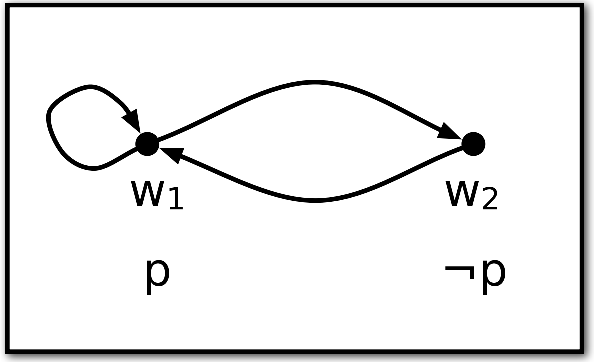
- The set FORM of well-formed formulas of propositional modal logic is the smallest set such that:
  - All propositional variables are in FORM
  - If A, B are in FORM, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$, $\Box A$, $\Diamond A$

Model Structures

- Model structures: $M = (W, R, V)$
  - W is a non-empty set (of "possible worlds")
  - $R \subseteq W \times W$ is an accessibility relation on W
  - V is value assignment function, which assigns each propositional constant a function $W \rightarrow \{0,1\}$
- For $V(p)(w)$ we also write $V_w(p)$ or $V_{M,w}(p)$.

Model Structures – Example

- M = (U, R, V)
  - U = {w_1, w_2}
  - R = {(w_1,w_1), (w_1,w_2), (w_2,w_1)}
  - V(p) = {(w_1, 1), (w_2, 0)}



Interpretation

$[\![p]\!]^{M,w} = 1$ iff $V_{M,w}(p) = 1$

$[\![\neg\varphi]\!]^{M,w} = 1$ iff $[\![\varphi]\!]^{M,w} = 0$

$[\![\varphi \wedge \psi]\!]^{M,w} = 1$ iff $[\![\varphi]\!]^{M,w} = 1$ and $[\![\psi]\!]^{M,w} = 1$

$[\![\varphi \vee \psi]\!]^{M,w} = 1$ iff $[\![\varphi]\!]^{M,w} = 1$ or $[\![\psi]\!]^{M,w} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,w} = 1$ iff $[\![\varphi]\!]^{M,w} = 0$ or $[\![\psi]\!]^{M,w} = 1$

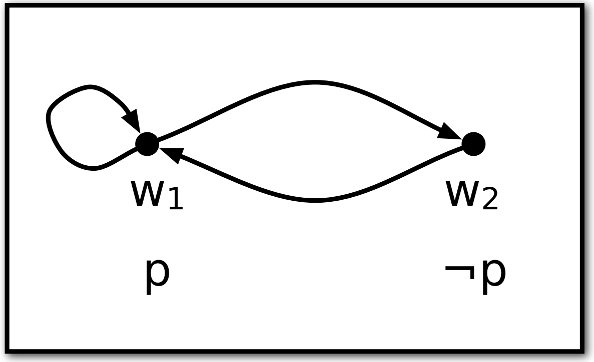
$[\![\varphi \leftrightarrow \psi]\!]^{M,w} = 1$ iff $[\![\varphi]\!]^{M,w} = [\![\psi]\!]^{M,w}$

$[\![\Diamond\varphi]\!]^{M,w} = 1$ iff there is a $w' \in W$ such that $R(w, w')$ and $[\![\varphi]\!]^{M,w'} = 1$

$[\![\Box\varphi]\!]^{M,w} = 1$ iff for all $w' \in W$ such that $R(w, w')$, $[\![\varphi]\!]^{M,w'} = 1$

An Example

- M = (U, R, V)
  - U = $\{w_1, w_2\}$
  - R = $\{(w_1,w_1), (w_1,w_2), (w_2,w_1)\}$
  - V(p) = $\{(w_1, 1), (w_2, 0)\}$



- Which of the following formulae are true in w_1, w_2, or the whole model?

(1) $\Box p \rightarrow \Box\Box p$

(2) $\neg\Box p$

(3) $p \rightarrow \Box\Diamond p$

Some Logical Laws

- The following formulae are true for every model structure and possible world.

(1) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

(2) $\Diamond A \leftrightarrow \neg\Box\neg A$

(3) $\Box A \leftrightarrow \neg\Diamond\neg A$

Some Logical Laws

- The following formulae are not true in all model structures, but true in model structures that have a certain structure.

  (1) $\Box A \rightarrow A$

  (2) $A \rightarrow \Diamond A$

  (3) $\Box A \rightarrow \Box\Box A$

  (4) $\Diamond\Diamond A \rightarrow \Diamond A$

  (5) $A \rightarrow \Box\Diamond A$

  (6) $\Diamond A \rightarrow \Box\Diamond A$

- (1) and (2) are true if R is reflexive (“System T”)
- (3) and (4) are true if R is transitive (“S4”)
- (5) and (6) are true if R is also symmetric (“S5”)

Propositional Tense Logic

- Tense logic is a variant of modal logic that deals with time rather than necessity.
  - Usually depends on an accessibility relation that is linear.
- Analogs of □ and ◇:
  - **G**φ – is always **going** to be the case
  - **H**φ – it **has** always been the case
  - **F**φ – it will be at some stage in the **future** be the case that φ
  - **P**φ – it was be at some stage in the **past** the case that φ

Intensional Predicate Logic

- Straightforward syntax:
  - first-order formulae, plus □ and ◇
- Extend model structures with possible worlds
- Interpretation of terms – various options
  - Should every possible world have its own domain?
    - … or should we assume one fixed domain for every world?
  - Should we interpret constants as individuals: $V(c) \in U$?
    - … or as individual concepts: $V(c): W \rightarrow U$?
- (see Gamut 1991, Vol 2, Chapter 3)

Model Structures

- Model structures: $M = (U, W, R, V)$
  - U is a non-empty set (the "universe")
  - W is a non-empty set (of "possible worlds")
    - $U \cap W = \varnothing$
  - R is a binary relation on W (the "accessibility relation")
  - V is value assignment function for non-logical constants
    - $V(a): W \rightarrow U_M$ for individual constants
    - $V(R): W \rightarrow U^n$ for n-place predicate symbols
- Assignment function for variables: $g: VAR \rightarrow U_M$

Interpretation of Terms

- $[\![c]\!]^{M,w,g} = V_M(w)(c)$ if c is a constant
- $[\![x]\!]^{M,w,g} = g(x)$ if x is a variable

- Note: here we interpret constants as “individual concepts”
  - a constant can refer to different individuals in different worlds

Interpretation of Formulae

$[\![R(t_1,\ldots,t_n)]\!]^{M,w,g} = 1$ iff $([\![t_1]\!]^{M,w,g}, \ldots, [\![t_n]\!]^{M,w,g}) \in V_M(w)(R)$

$[\![s = t]\!]^{M,w,g} = 1$ iff $[\![s]\!]^{M,w,g} = [\![t]\!]^{M,w,g}$

$[\![\neg\varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g} = 0$

$[\![\varphi \wedge \varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g} = 1$ and $[\![\varphi]\!]^{M,w,g} = 1$

$[\![\varphi \vee \varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g} = 1$ or $[\![\varphi]\!]^{M,w,g} = 1$

$[\![\varphi \rightarrow \varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g} = 0$ or $[\![\varphi]\!]^{M,w,g} = 1$

$[\![\exists x\varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g[a/x]} = 1$ for some $a \in U_M$

$[\![\forall x\varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w,g[a/x]} = 1$ for every $a \in U_M$

$[\![\Box\varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w',g} = 1$ for every w' with $R(w, w')$

$[\![\Diamond\varphi]\!]^{M,w,g} = 1$ iff $[\![\varphi]\!]^{M,w',g} = 1$ for at least one w' with $R(w, w')$

Substitutability, revisited

(1) Necessarily, the American president is the Supreme Commander of the Armed Forces.

- $\Box(\exists x(\forall y(P(y) \leftrightarrow x = y) \land C(x))$

(2) Barack Obama is the American president.

- $\exists x(\forall y(P(y) \leftrightarrow x = y) \land x = o)$

(3) Necessarily, Barack Obama is the Supreme Commander of the Armed Forces.

- $\Box C(o)$

P(x) – *x is the American president*

C(x) – *x is the Supreme Commander of the Armed Forces*

o – *Barack Obama*

Substitutability, revisited

- $[\![\exists x(\forall y(P(y) \leftrightarrow x = y) \wedge C(x))]\!]^{M,w,g} = 1$
  - iff exists $a \in U_M$: $a \in V_M(P)$ and $|V_M(P)| = 1$ and $a \in V_M(C)$
- $[\![\Box\exists x(\forall y(P(y) \leftrightarrow x = y) \wedge C(x))]\!]^{M,w,g} = 1$
  - iff for all w' such that R(w,w'), …
- $[\![\Box C(o)]\!]^{M,w,g} = 1$
  - iff for all w' such that R(w,w'), $V_M(o) \in V_M(C)$

	w_1	w_2	w_3	w_4	…
P	{a}	{a}	{a}	{b}	…
C	{a}	{a}	{a}	{b}	…
o	b	b	b	b	…

$R = W \times W$

Literature

- Gamut (1991). *Logic, Language, and Meaning*, Vol. 2, Chapters 2 and 3