Semantic Theory
Predicate Logic

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Overview

- A Reminder: First-Order Predicate Logic (FOL)
  - syntax & semantics
  - truth, satisfiability, validity, entailment
- Representing natural language meaning

Why Logic?

- Logic supports precise, consistent and controlled meaning representation via truth-conditional interpretation.
- Logic provides deduction systems to model inference processes, controlled by a formal entailment concept.
- Logic supports uniform modelling of the semantic composition process (⇒ type theory)

Predicate Logic – Vocabulary

- Non-logical expressions:
  - Individual constants: CON
  - n-place relation constants: PRED^n, for all $n \geq 0$
- Individual variables: VAR

Predicate Logic – Syntax

- Terms: TERM = VAR ∪ CON
- Atomic formulas:
  - $R(t_1, \ldots, t_n)$ for $R \in PRED^n$ and $t_1, \ldots, t_n \in TERM$
  - $s = t$ for $s, t \in TERM$
- Well-formed formulas: the smallest set FORM such that
  - All atomic formulas are in FORM
  - If φ, ψ are in FORM, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are in FORM
  - If x is individual variable, and φ is in FORM, then $\forall x\varphi$ and $\exists x\varphi$ are in FORM.

Scope

- If $\forall x\varphi$ ($\exists x\varphi$) is a subformula of a formula ψ, then we call φ the scope of this occurrence of $\forall x$ ($\exists x$) in ψ.
- We distinguish distinct occurrences of quantifiers as there are formulae like $\forall xA(x) \land \forall xB(x)$.
- Examples:
  - $\exists x\underline{(\forall y\underline{(T(y) \leftrightarrow x=y)} \land F(x))}$
  - $\forall x\underline{A(x)} \land \forall x\underline{B(x)}$

Free and Bound Variables

- An occurrence of a variable x in a formula φ is said to be free in φ if this occurrence of x does not fall within the scope of a quantifier ∀x or ∃x in φ.
- If ∀xψ (or ∃xψ) is a subformula of φ and x is free in ψ, then this occurrence of x is said to be bound by this occurrence of the quantifier ∀x (or ∃x).
- Examples:
  - $\forall x(A(x) \wedge B(x))$ – x occurs bound in B(x)
  - $\forall x\, A(x) \wedge B(x)$ – x occurs free in B(x)
- A sentence is a formula without free variables.

Free Variables

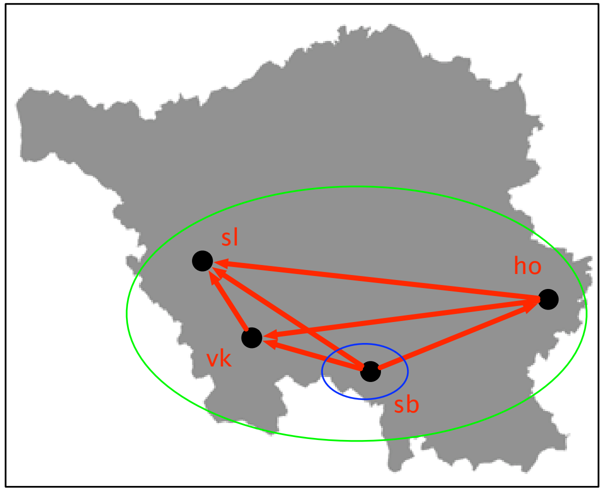
- Let VAR(φ) be the set of all variables of some formula φ.
- We can define the set FV(φ) of free variables in φ recursively as follows:

$$
\begin{aligned}
FV(\varphi) &= VAR(\varphi)\text{, if } \varphi \text{ atomic formular}\\
FV(\neg\varphi) &= FV(\varphi)\\
FV(\varphi \land \psi) &= FV(\varphi) \cup FV(\psi)\\
FV(\varphi \lor \psi) &= FV(\varphi) \cup FV(\psi)\\
&\dots\\
FV(\exists x \Phi) &= FV(\Phi) - \{x\}\\
FV(\forall x \Phi) &= FV(\Phi) - \{x\}
\end{aligned}
$$

Predicate Logic – Semantics

- Expressions of Predicate Logic are interpreted relative to model structures and variable assignments.
- Model structure: $M = (U_M, V_M)$
  - U_M is non-empty set – the “universe”
  - V_M is an interpretation function (for non-logical symbols) assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:
    - $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
    - $V_M(c) \in U_M$ if c is an individual constant
- Assignment function for variables $g: VAR \rightarrow U_M$

A model of Saarland



- $M = (U_M, V_M)$
- $U_M = \{sl, vk, ho, sb\}$
- V_M defined by:
  - V_M(saarbrücken) = sb
  - V_M(saarluis) = sl
  - V_M(völkingen) = vk
  - V_M(homburg) = ho
  - V_M(largerthan) = { (sb, sl), (sb, vk), (sb, ho), (vk, sl), … }
  - V_M(town) = { sl, vk, ho, sb }
  - V_M(capital) = { sb }

Interpretation of terms

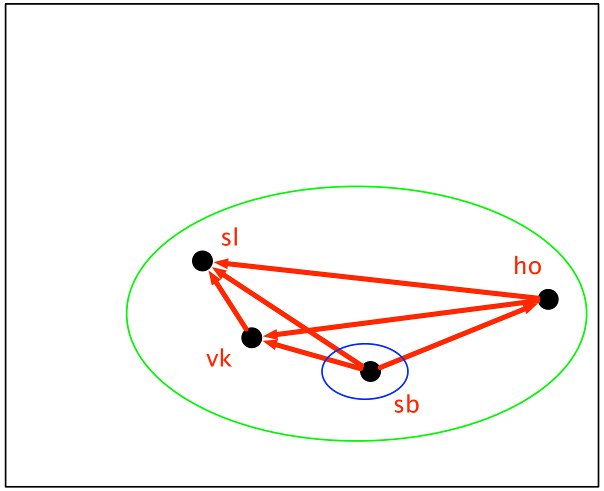
- Interpretation of terms with respect to a model structure M and a variable assignment g:
  - $[\![\alpha]\!]^{M,g} = V_M(\alpha)$, if α is an individual constant
  - $[\![\alpha]\!]^{M,g} = g(\alpha)$, if α is a variable

Interpretation of atomic formulas

- Terms: TERM = VAR ∪ CON
- Atomic formulas:
  - $R(t_1,\ldots, t_n)$ for $R \in PRED^n$ and $t_1, \ldots, t_n \in TERM$
  - $s = t$ for $s, t \in TERM$
- Interpretation of (atomic) formulas with respect to a model structure M and variable assignment g:
  - $[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $([\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g}) \in V_M(R)$
  - $[\![s = t]\!]^{M,g} = 1$ iff $[\![s]\!]^{M,g} = [\![t]\!]^{M,g}$

Interpretation of atomic formulas

- $[\![$ largerthan(saarbrücken, völklingen) $]\!]^{M,g} = 1$
  - iff $([\![$saarbrücken$]\!]^{M,g}$, $[\![$völklingen$]\!]^{M,g}) \in V_M$(largerthan)
  - iff (sb, vk) $\in V_M$(largerthan)



Interpretation of connectives

- Connectives are truth-functional: the truth-value of a complex expession is completely determined by the truth-values of their subformulas.

$$[\![\neg\varphi]\!]^{M,g} = 1 \text{ iff } [\![\varphi]\!]^{M,g} = 0$$

$$[\![\varphi \wedge \psi]\!]^{M,g} = 1 \text{ iff } [\![\varphi]\!]^{M,g} = 1 \text{ and } [\![\psi]\!]^{M,g} = 1$$

$$[\![\varphi \vee \psi]\!]^{M,g} = 1 \text{ iff } [\![\varphi]\!]^{M,g} = 1 \text{ or } [\![\psi]\!]^{M,g} = 1$$

$$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1 \text{ iff } [\![\varphi]\!]^{M,g} = 0 \text{ or } [\![\psi]\!]^{M,g} = 1$$

$$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1 \text{ iff } [\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$$

Interpretation of quantifiers

- Interpretation of quantified formulas ...
  - $[\![\exists x A(x)]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $d \in [\![A]\!]^{M,g}$
  - $[\![\forall x A(x)]\!]^{M,g} = 1$ iff for every $d \in U_M$, $d \in [\![A]\!]^{M,g}$
- Interpretation of formulas with respect to a model structure M and variable assignment g:
  - $[\![\exists x \varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$
  - $[\![\forall x \varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$
- $g[x/d]$ is the variable assignment which is identical to g except that it assigns the individual d to variable x.
  - $g[x/d](y) = d$ if $x = y$,
  - $g[x/d](y) = g(y)$ if $x \neq y$

Variable assignments

- g[x/d] is the variable assignment which is identical to g except that it assigns the individual d to variable x.

	x	y	z	u	…
g	a	b	c	d	…
g[x/a]	a	b	c	d	…
g[y/a]	a	a	c	d	…
g[y/g(z)]	a	c	c	d	…
g[y/a][u/a]	a	a	c	a	…
g[y/a][y/b]	a	b	c	d	…

Bill does’nt know every student

- $[\![\neg\forall x(\text{student}(x) \rightarrow \text{know}(\text{bill}, x))]\!]^{M,\,g} = 1$ iff …

[whiteboard]

Interpretation of Predicate Logic

- Interpretation of formulas with respect to a model structure M and variable assignment g:

$[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $([\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g}) \in V_M(R)$

$[\![s = t]\!]^{M,g} = 1$ iff $[\![s]\!]^{M,g} = [\![t]\!]^{M,g}$

$[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

$[\![\exists x\varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$

$[\![\forall x\varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$

- g[x/d] is the variable assignment which is identical to g except that it assigns the individual d to variable x.

Truth &al.

- A formula Φ is true in a model structure M iff $[\![\Phi]\!]^{M,g} = 1$ for every variable assignment g.
- A formula Φ is valid ($\models \Phi$) iff Φ is true in all model structures.
- A formula Φ is satisfiable iff there is at least one model structure M such that φ is true in M.

Truth &al.

- A set of formulas Γ is (simultaneously) satisfiable iff there is a model structure M s.t. every formula in Γ is true in M.
  - We also say that M satisfies Γ, or M is a model of Γ
- Γ is contradictory if Γ is not satisfiable.
- A set of formulas Γ entails a formula Φ ($\Gamma \models \Phi$) iff Φ is true in every model structure that satisfies Γ.
- Two formulas Φ, Ψ are eqivalent if $[\![\Phi]\!]^{M,g} = [\![\Psi]\!]^{M,g}$ for all model structures M and assignments g.

True? Satisfiable? Valid?

(1) $\forall x F(x) \rightarrow \exists x F(x)$

(2) $\exists x (F(x) \wedge \neg F(x))$

(3) $\exists x\, F(x) \wedge \neg F(x)$

Some logical laws (connectives)

- Distributive Laws
  - $(A \lor (B \land C)) \Leftrightarrow ((A \lor B) \land (A \lor C))$
  - $(A \land (B \lor C)) \Leftrightarrow ((A \land B) \lor (A \land C))$
- DeMorgan‘s Laws
  - $\neg(A \lor B) \Leftrightarrow (\neg A \land \neg B)$
  - $\neg(A \land B) \Leftrightarrow (\neg A \lor \neg B)$
- [...]

Some logical laws (quantifiers)

- Quantifier negation:
  - $\neg\forall x\Phi \Leftrightarrow \exists x\neg\Phi$
- Quantifier distribution:
  - $\forall x(\Phi \wedge \Psi) \Leftrightarrow \forall x\Phi \wedge \forall x\Psi$
  - $\exists x(\Phi \vee \Psi) \Leftrightarrow \exists x\Phi \vee \exists x\Psi$
- Quantifier (in-)dependence
  - $\forall x\ \forall y\ \Phi \Leftrightarrow \forall y\ \forall x\ \Phi$
  - $\exists x\ \exists y\ \Phi \Leftrightarrow \exists y\ \exists x\ \Phi$
  - $\exists x\ \forall y\ \Phi \Rightarrow \forall y\ \exists x\ \Phi$ (but not vice versa)

Some logical laws (quantifiers)

- Quantifier movement
  - $\Phi \rightarrow \forall x \Psi \quad \Leftrightarrow \quad \forall x(\Phi \rightarrow \Psi)$
  - $\Phi \rightarrow \exists x \Psi \quad \Leftrightarrow \quad \exists x(\Phi \rightarrow \Psi)$
  - $\forall x \Psi \rightarrow \Phi \quad \Leftrightarrow \quad \forall x(\Psi \rightarrow \Phi)$
  - $\exists x \Psi \rightarrow \Phi \quad \Leftrightarrow \quad \exists x(\Psi \rightarrow \Phi)$
- … provided that x does not occur free in Φ

Representing (sentence) meaning

- The meaning of a natural language sentence S can be approximated by the truth-conditions of S.
- Logical expressions can be used to represent the truth-conditions of natural language sentences.

Talking about students

(1) *Bill is a student*

$\Rightarrow$ student(bill)

(2) *Bill reads an interesting book*

$\Rightarrow \exists x((\text{book}(x) \wedge \text{interesting}(x)) \wedge \text{read}(\text{bill}, x))$

(3) *Bill reads every interesting book*

$\Rightarrow \forall x((\text{book}(x) \wedge \text{interesting}(x)) \rightarrow \text{read}(\text{bill}, x))$

(4) *Not all students passed [the exam]*

$\Rightarrow \neg\forall x(\text{student}(x) \rightarrow \text{pass}(x))$

(5) *Only Bill flunked*.

$\Rightarrow \neg\text{pass}(\text{bill}) \wedge \forall x(\neg\text{pass}(x) \rightarrow x = \text{bill})$

More Examples

(1) *Bill is annoyed if someone is noisy*

⇒ $\exists x\ noisy(x) \rightarrow annoyed(bill)$

(2) *Bill is annoyed only if somebody is noisy*

⇒ $annoyed(bill) \rightarrow \exists x\ noisy(x)$

(3) *Although nobody is noisy, Bill is annoyed.*

⇒ $\neg\exists x\ noisy(x) \wedge annoyed(bill)$

(4) *Everyone who lives in Saarbrücken loves it*

⇒ $\forall x(live\text{-}in(x, saarbrücken) \rightarrow love(x, saarbrücken))$

(5) *No one answered every question*

⇒ $\neg\exists x\ \forall y(question(y) \rightarrow answer(x, y))$