Introduction to Typed Lambda Calculus

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(1) **Typed Lambda Calculi (TLCs)**

- Developed starting with Church and Curry in 1930's
- Can be viewed model-theoretically (Henkin-Montague perspective) or proof-theoretically (Curry-Howard perspective). For now we focus on the former.
- Underlies higher-order logic (HOL), widely used in semantics for formalizing theories of meaning (and in LG for formalizing phenogrammar).
- A TLC is specified by giving its *types*, its *terms*, and an *equivalence relation* on the terms.
- There are different kinds of TLCs, depending on what kind of propositional logic its type system is based on.
- The TLC that underlies HOL, called *positive* TLC, is based on *positive intuitionistic propositional logic* (PIPL), which lacks false, negation and disjunction.

(2) **Types of Positive TLC**

a. There are some *basic* types. For concreteness, in this handout we use two basic types motivated by NL semantics:

 p (for *propositions*, the kind of meanings expressed by utterances of declarative sentences)

 e (for *entities*, the kind of things that can be meanings of names (just for now assuming a direct reference theory of names)

b. T is a type;
c. if A and B are types, so is $A \wedge B$; and
d. if A and B are types, so is $A \to B$.

(3) **Terms of Positive TLC**

Note: we write '$\vdash a : A$' to mean term a is of type A.

a. There are some *nonlogical constants*. In the typical application to NL semantics, these are interpreted as word meanings, e.g.:

$\vdash \mathsf{fido} : \mathrm{e}$
$\vdash \mathsf{bark} : \mathrm{e} \to \mathrm{p}$
$\vdash \mathsf{bite} : (\mathrm{e} \wedge \mathrm{e}) \to \mathrm{p}$
$\vdash \mathsf{give} : (\mathrm{e} \wedge \mathrm{e} \wedge \mathrm{e}) \to \mathrm{p}$
$\vdash \mathsf{believe} : (\mathrm{e} \wedge \mathrm{p}) \to \mathrm{p}$

b. There is a *logical constant* $\vdash * : \mathrm{T}$. In the application to NL semantics, this is interpreted as the vacuous meaning.
c. For each type A there are variables $\vdash x_i^A : A$ $(i \in \omega)$
d. If $\vdash a : A$ and $\vdash b : B$, then $\vdash (a, b) : A \wedge B$;
e. If $\vdash a : A \wedge B$, then $\vdash \pi(a) : A$;
f. If $\vdash a : A \wedge B$, then $\vdash \pi'(a) : B$;
g. If $\vdash f : A \to B$ and $\vdash a : A$, then $\vdash \mathsf{app}(f, a) : B$;[1]
h. If $\vdash x : A$ is a variable and $\vdash b : B$, then $\vdash \lambda_x.b : A \to B$.

(4) **Positive TLC Term Equivalences**

Here t, a, b, p, and f are metavariables ranging over terms.

a. Equivalences for the term constructors:
  i. $t \equiv *$ (for t a term of type T);
  ii. $\pi(a, b) \equiv a$;
  iii. $\pi'(a, b) \equiv b$; and
  iv. $(\pi(p), \pi'(p)) \equiv p$

[1] $\mathsf{app}(f, a)$ is usually abbreviated to $(f\ a)$.

b. Equivalences for the variable binder ('lambda conversion')[2]

(α) $\lambda_x.b \equiv \lambda_y.[y/x]b$;

(β) $(\lambda_x.b\ a) \equiv [a/x]b$; and

(η) $\lambda_x.(f\ x) \equiv f$, provided x is not free in f.

c. Equivalences of Equational Reasoning

(ρ) $a \equiv a$

(σ) If $a \equiv a'$, then $a' \equiv a$.

(τ) If $a \equiv a'$ and $a' \equiv a''$, then $a \equiv a''$.

(ξ) If $b \equiv b'$, then $\lambda_x.b \equiv \lambda_x.b'$.

(μ) If $f \equiv f'$ and $a \equiv a'$, then $(f\ a) \equiv (f'\ a')$.

(5) **The Henkin-Montague Perspective**

A **(set-theoretic) interpretation** I of a positive TLC assigns to to each type A a set $I(A)$ and to each constant $\vdash a : A$ a member $I(a)$ of $I(A)$, subject to the following constraints:

a. $I(\mathrm{T})$ is a singleton;

b. $I(A \wedge B) = I(A) \times I(B)$;

c. $I(A \to B) \subseteq I(A) \to I(B)$.[3]

(6) **Assignments**

An **assignment** relative to an interpretation is a function that maps each member of a set of variables to a member of the set that interprets the variable's type.

(7) **Extending an Interpretation Relative to an Assignment**

Given an assignment α relative to an interpretation I, there is a unique extension of I, denoted by I_α, that assigns interpretations to all terms, such that:

a. For each variable x, $I_\alpha(x) = \alpha(x)$;

b. for each constant a, $I_\alpha(a) = I(a)$;

c. if $\vdash a : A$ and $\vdash b : B$, then $I_\alpha((a, b))$ is $\langle I_\alpha(a), I_\alpha(b)\rangle$;

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[2]The notation '$[a/x]b$' means the term resulting from substituition in b of all free occurrences of $x : A$ by $a : A$. This presupposes no free variable occurences in a become bound as a result of the substitution.

[3]As in Henkin 1950, the set inclusion in clause (3) can be proper, as long as there are enough functions to interpret all functional terms.

d. if $\vdash p : A \wedge B$, then $I_\alpha(\pi(p))$ is the first component (= projection onto $I(A)$) of $I_\alpha(p)$; and $I_\alpha(\pi'(p))$ is the second component (= projection onto $I(B)$) of $I_\alpha(p)$;

e. if $\vdash f : A \to B$ and $\vdash a : A$, then $I_\alpha((f\ a)) = (I_\alpha(f))(I_\alpha(a))$; and

f. if $\vdash b : B$, then $I_\alpha(\lambda_{x \in A}.b)$ is the function from $I(A)$ to $I(B)$ that maps each $s \in I(A)$ to $I_\beta(b)$, where β is the assignment that coincides with α except that $\beta(x) = s$.

(8) **Observations about Interpretations**

- Two terms $\vdash a : A$ and $\vdash b : B$ of positive TLC are term-equivalent iff $A = B$ and, for any intepretation I and any assignment α relative to I, $I_\alpha(a) = I_\alpha(b)$.
- Another way of stating the preceding is to say that term equivalence (viewed as an equational proof system) is sound and complete for the class of interpretations described in (5-7).
- For any term a, $I_\alpha(a)$ depends only on the restriction of α to the free variables of a.
- In particular, if a is a closed (i.e. has no free variables), then $I_\alpha(a)$ is independent of α so we can simply write $I(a)$.
- Thus, an interpretation for the basic types and constants extends uniquely to all types and all closed terms.

(9) **Sequent-Style ND with Proof Terms**

- We review a style of ND equipped to analyze not just provability, but also proofs.
- We illustrate how this works for PIPL, starting from the (term-free) sequent-style ND for PIPL already introduced.
- We'll see that in addition to (or at the same time as) being thought of as denoting elements of models, TLC terms can also be thought of as notations for proofs.
- This idea was first articulated by Curry (1934, 1958), then elaborated by Howard (1969 [1980]), Tait (1967), etc..
- Soon, we'll use this kind of ND for phenos and meanings in linguistic derivations.

(10) **Preliminary Definitions**

1. a (TLC) term is called **closed** if it has no free variables;
2. a closed term is called a **combinator** if it contains no nonlogical constants;
3. a type is said to be **inhabited** if there is a closed term of that type.

(11) **Curry-Howard Correspondence (1/2)**

- Types are (the same thing as) formulas.
- Type constructors are logical connectives.
- (Equivalence classes of) terms are proofs.
- The free variables of a term are the undischarged hypotheses on which the proof depends.
- The nonlogical constants of a term are the nonlogical axioms used in the proof.
- A type A is a theorem iff it is inhabited.
- A type is a pure theorem (requires no nonlogical axioms to prove it) iff it is inhabited by a combinator.

(12) **Curry-Howard Correspondence (2/2)**

- Application corresponds to modus ponens.
- Abstraction corresponds to hypothetical proof (discharge of hypothesis).
- Pairing corresponds to conjunction introduction.
- Projections correspond to conjunction eliminations.
- Identification of free variables corresponds to collapsing of duplicate hypotheses (contraction).
- Vacuous abstraction corresponds to discharge of a nonexistent hypothesis (weakening).

(13) **Notation for Sequent-Style ND with Proof Terms**

Judgments are of the form $\Gamma \vdash a : A$, read 'a is a proof of A with hypotheses Γ', where

1. A is a formula (= type)
2. a is a term (= proof)
3. Γ, the **context** of the judgment, is a set of variable/formula pairs of the form $x : A$, with a distinct variable in each pair.

(14) **Axiom Schemas**

Hypotheses:

$$x : A \vdash x : A$$

(x a variable of type A)

Nonlogical Axioms:

$$\vdash a : A$$

(a a nonlogical constant of type A)

Logical Axiom:

$$\vdash * : \mathrm{T}$$

(15) **Rule Schemas for Implication**

→-Elimination or Modus Ponens:

$$\frac{\Gamma \vdash f : A \to B \qquad \Delta \vdash a : A}{\Gamma, \Delta \vdash (f\ a) : B} \to\mathrm{E}$$

This presupposes no variable occurs in both Γ and Δ.

→-Introduction or Hypothetical Proof:

$$\frac{x : A, \Gamma \vdash b : B}{\Gamma \vdash \lambda_x . b : A \to B} \to\mathrm{I}$$

(16) **Other Rule Schemas**

There are also rules schemas (which we will not need) for:

- pairing/conjunction introduction
- projections/conjunction elimination
- identifying variables/contraction
- useless hypotheses/weakening

For details see e.g. John C. Mitchell and Philip J. Scott (1989) "Typed lambda models and cartesian closed categories", **Contemporary Mathematics** 92:301-316.