Overview of Higher-Order Logic

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(1) **Extending TLC to HOL (Church 1940)**

- Start with a TLC.
- Add a type t (Church's o) for truth values.
- Add equality symbols $=_A$ for all types.
- Re-express the TLC term equivalences as object-language axioms about equality.
- *Define* the usual (term-level, not type-level) classical logical connectives and quantifiers in terms of λ and equality.

(2) **A Logic Defined in this Way:**

- has all the term equalities expected in TLC ('lambda conversion')
- has all the (term-level) theorems of classical FOL
- allows quantification over variables of all types.

(3) **Historical Synopsis of Classical HOL**

- Henkin(1947/1950) reaxiomatized Church's (1940) Simple Theory of Types
  - added a key axiom (*Truth-Value Extensionality*) identifying identity of truth values with bi-implication, and
  - proved completeness relative to the class of set-theoretic models that bear his name;
- Gallin (1975) showed that Henkin's HOL with two basic types (besides t) instead of just one was equivalent to Montague's IL

- Groenendijk and Stokhof (1980s) started using Ty2 instead of IL for NL semantics.
- Lambek and Scott (1986) added *subtyping* (analogous to the Axiom of Separation in set theory), and allowed a wider class of (not necessarily set-theoretic) models (*toposes*).

(4) **Our HOL**

- It has everything positive TLC has (including the type constructors T (interpreted as a singleton) and $A \wedge B$ (interpreted as cartesian product)
- Our $A \rightarrow B$ is what Montague wrote as $\langle A, B \rangle$.
- We'll use slightly different basic types than Montague did.
- We will have **subtyping** (Lambek and Scott 1986), to be described soon.

(5) **HOL: a Closer Look**

a. We start with a (PIPL-based) TLC and add a new type t. (This is part of the logic, not a basic type added at the user's discretion.)

b. Terms of type t are called **formulas**. (Note: 'formula' is now ambiguous between 'type' and 'term of type t'.)

c. Axioms will ensure that $I(\mathrm{t})$ has exactly two members (called **truth values**), for any interpretation I.

d. For each type A, we have a constant $=_A$: $(A \wedge A) \rightarrow \mathrm{t}$, written infix $(a = b)$. The type subscript is usually omitted.

e. $I(=_A)$ is the identity relation on $I(A)$.

(6) **Classical Connectives and Quantifiers are Definable**
Here ϕ is a metavariable over formulas, x is a variable of type A, and s, t are variables of type t:

a. $\mathsf{true} =_{\mathrm{def}} * = *$

b. $\forall_x.\phi =_{\mathrm{def}} \lambda_x.\phi = \lambda_x.\mathsf{true}$

c. $\mathsf{false} =_{\mathrm{def}} \forall_t.t$

d. $\phi \wedge \psi =_{\mathrm{def}} (\phi, \psi) = (\mathsf{true}, \mathsf{true})$

e. $\phi \rightarrow \psi =_{\mathrm{def}} \phi = (\phi \wedge \psi)$

f. $\phi \leftrightarrow \psi =_{\mathrm{def}} (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$

g. $\sim \phi =_{\text{def}} \phi \rightarrow \mathsf{false}$
h. $\phi \vee \psi =_{\text{def}} \sim [(\sim \phi) \wedge (\sim \psi)]$
i. $\exists_x.\phi =_{\text{def}} \sim \forall_x. \sim \phi$

(7) **Numerous Options for Axiomatizing HOL**

- Gallin (Ty2, 1975) (essentially follows Henkin 1950);
- Carpenter (1997) (essentially follows Andrews 1986);
- Lambek and Scott (1986) have subtyping (see below), and the option of 'going intuitionistic' (dropping Excluded Middle from the term logic)
- We'll remain agnostic about how to axiomatize HOL, and just mention some useful rules and axioms (or theorems, depending on the choice of axiomatization).
- We write $\vdash \phi$ to mean 'ϕ is provable in HOL'. (Note that '$\vdash$' is the same symbol used in typing judgments.)

(8) **Equality is an Equivalence Relation**

a. $\vdash a = a$ (reflexivity)
b. if $\vdash a = b$, then $\vdash b = a$ (symmetry)
c. If $\vdash a = b$ and $\vdash b = c$, then $\vdash a = c$ (transitivity)

(9) **Rules for Substitution of Equals**

a. if $\vdash a = c$ and $\vdash b = d$, then $\vdash (a, b) = (c, d)$
b. if $\vdash f = g$ and $\vdash a = b$, then $\vdash f(a) = g(b)$
c. if $\vdash a = b$, then $\vdash \lambda_x.a = \lambda_x.b$

(10) **Axioms for Cartesian Products**

a. if $\vdash a : \mathrm{T}$, then $\vdash a = *$
b. $\vdash \pi(a, b) = a$
c. $\vdash \pi'(a, b) = b$
d. $\vdash (\pi(c), \pi'(c)) = c$

(11) **Axioms for Lambda Conversion**

a. $\vdash \lambda_{x \in A}.b = \lambda_{y \in A}.[y/x]b$ (rule α)
b. $\vdash ((\lambda_{x \in A}.b)\ a) = [a/x]b$ (rule β)
c. if $\vdash f : A \to B$ and x is not free in f, then $\vdash (\lambda_{x \in A}.f\ x) = f$ (rule η)

(12) **Axiom of Excluded Middle**

$\vdash \forall_t.t \vee \sim t$

(13) **Axiom of Nondegeneracy**

$\vdash \sim (\mathsf{true} = \mathsf{false})$

(14) **Axioms for Equality of Truth Values**

a. $\vdash \phi = (\phi = \mathsf{true})$
b. If $\vdash \phi$ and $\vdash \phi = \psi$, then $\vdash \psi$
c. $\vdash \phi$ iff $\vdash \phi = \mathsf{true}$
d. $\vdash \forall_{s,t}.(s \leftrightarrow t) \to (s = t)$ (Truth-Value Extensionality)

(15) **Motivation for Subtypes**

- Standard HOL has no way to say A is a *subtype* of B.
- In an intepretation I, this should mean $I(A) \subseteq I(B)$.
- Example: we will want to *define* the type w (worlds) as a certain subtype of the type p $\to$ t of sets of propositions (namely the ones which are *maximal* and *consistent*).

(16) **Subtypes (after Lambek and Scott 1986)**

If A is a type and a an A-predicate (i.e. a closed term of type $A \to$ t), then

- A_a is a type
- embed_a is a term of type $A_a \to A$; and
- Axioms:

$$\vdash \forall_{y,z \in A_a}.((\mathsf{embed}_a\ y) = (\mathsf{embed}_a\ z)) \to y = z)$$
$$\vdash \forall_{x \in A}.(a\ x) \leftrightarrow \exists_{y \in A_a}.x = (\mathsf{embed}_a\ y)$$

(17) **What Subtypes Mean in an Interpretation I**

- $I(a)$ is a function from $I(A)$ to truth values
- $I(\mathsf{embed}_a)$ is a one-to-one function from $I(A_a)$ to $I(A)$
- the members of $I(A)$ that $I(a)$ maps to $I(\mathsf{true})$ are the ones that are embedded images of members of $I(A_a)$.

So $I(\mathsf{embed}_a)$ is the function that embeds into $I(A)$ the subset whose characteristic function is $I(a)$.