HYPERINTENSIONAL SEMANTICS

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TENTATIVE COURSE OVERVIEW (DOUBTLESS OVERLY AMBITIOUS)

Day One

Lecture 1: Introduction and Motivation

Lecture 2: Problems with Standard Possible-Worlds Semantics

Day Two

Lecture 3: Soft Actualism Defined and Algebraicized

Lecture 4: The Positive Typed Lambda Calculus

Day Three

Lecture 5: Higher Order Logic with Subtypes

Lecture 6: Hyperintensions and Entailment

Day Four

Lecture 7: Worlds, Extensions, and Equivalence

Lecture 8: Quantifiers and Modality

Day Five

Lecture 9: Questions

Lecture 10: Wrap-Up

LECTURE SEVEN:
WORLDS, EXTENSIONS, AND EQUIVALENCE

(1) **Goals of Lecture Seven**

- To extend our formal semantic theory to encompass a Soft-Actualist conception of **worlds**
- To further extend the theory to make clear what we mean by the **extension** of a hyperintension at a world
- To extend the notion of truth-conditional equivalence (mutual entailment) to a general notion of **equivalent hyperintensions**.

A HIGHER-ORDER
THEORY OF WORLDS

(2) **Worlds Revisited**

a. So far our theory doesn't mention worlds, but we need them:
   i. to define the extension of a hyperintension at a world
   ii. to analyze modality
   iii. to analyze counterfactuals, etc.

b. Following Soft Actualism, we treat worlds not as primitives, but rather as maximal consistent sets (ultrafilters) of propositions.

c. Fortunately, the predicate (of sets of propositions) of being an ultrafilter is HOL-definable.

d. So we can *define* the type World as the subtype of Prop $\supset$ Bool consisting of those sets of propositions which are ultrafilters (with respect to the entailment preorder on the set of propositions).

(3) **Worlds Defined**

We define World to be the type

$$(\mathrm{Prop} \supset \mathrm{Bool})_{\mathsf{u}}$$

where $\mathsf{u} : (\mathrm{Prop} \supset \mathrm{Bool}) \supset \mathrm{Bool}$ is the predicate on sets of propositions such that $\mathsf{u}(s)$ says of s that it is an ultrafilter.

But what exactly is u?

(4) **Worlds Defined (Continued)**

To see what term u must be, remember that, in a strict boolean preorder P, an ultrafilter is defined to be a subset s such that $\bot \notin s$, and for all $p, q \in P$:

a. if $p, q \in s$ then $p \sqcap q \in s$;
b. if $p \in s$ and $p \sqsubseteq q$, then $q \in s$; and
c. either $p \in s$ or $\neg p \in s$.

So we take u to be the term

$\lambda_s[(\sim s(\mathsf{Falsity})) \wedge \forall_{p,q}(\phi \wedge \psi \wedge \xi)]$, where:

a. ϕ is $(s(p) \wedge s(q)) \supset s(p \text{ } \mathsf{and'} \text{ } q)$
b. ψ is $(s(p) \wedge (p \models q)) \supset s(q)$
c. ξ is $s(p) \vee s(\mathsf{not'}(p))$.

(5) **A Technical, but Necessary, Point**

a. Recall that the general machinery for subtyping always provides, for any predicate $a : A \supset \text{Bool}$, a constant $\mathsf{embed}_a : A_a \supset A$ that is interpreted as the embedding function for the subset that has (the interpretation of) a as its characteristic function.

b. In the present case, where A is $\text{Prop} \supset \text{Bool}$, a is u, and the defined subtype is World ($= (\text{Prop} \supset \text{Bool})_{\mathsf{u}}$), the term for the embedding function is

$$\mathsf{embed}_{\mathsf{u}} : \text{World} \supset (\text{Prop} \supset \text{Bool})$$

c. We will have a practical application for this very soon.

(6) **How to Say a Proposition p is True at a World w**

a. In standard PWS, the way to say it is: $p(w)$.

b. That's because p is a set of worlds (the ones p is true at).

c. But in hyperintensional semantics, w is a set of propositions (the ones true at w; so, seemingly, the right way to say it is: $w(p)$.

d. But there's a minor glitch: $w(p)$ is an ill-typed term, since w is not of type Prop $\supset$ Bool, but rather of type World (= (Prop $\supset$ Bool)$_\mathsf{u}$).

e. This is fixed by using $\mathsf{embed}_\mathsf{u}(w)$ instead of w.

f. And so the right way to say p is true at w is: $\mathsf{embed}_\mathsf{u}(w)(p)$.

g. We will usually abbreviate this as $p@w$.

(7) **Are there Enough Worlds?**

a. Recall that in our metalanguage version of Soft Actualism, we relied crucially on Stone's Lemma.

b. This guarantees there are enough ultrafilters so that, for any two propositions p and q, if p does not entail q, then we can find an ultrafilter that has p, but not q, as a member.

c. Likewise, in our higher-order formalization of Soft Actualism, we need an object-language version of Stone's Lemma, or at least the special case of it where the strict boolean preorder in question is the set of propositions preordered by entailment.

d. We can just make this an axiom of our theory by brute force:

$\vdash \forall_{p,q}[(p \not\models q) \supset \exists_s(\mathsf{u}(s) \wedge s(p) \wedge \ \sim s(q))]$

e. Alternatively, there are various higher-order versions of the Axiom of Choice we could adopt, any of which would prove (7d).

(8) **Equivalent Propositions Revisited**

An immediate corollary of (7d) is:

$\vdash \forall_{p,q}[(p \equiv q) \leftrightarrow \ \forall_w(p@w \leftrightarrow \ q@w)]$

That is: two propositions are equivalent iff they are true in the same worlds.

(9) **More Consequences of Stone's Lemma**

Using the boolean preorder axioms for propositions (Lecture Six) together with Stone's Lemma, we can easily prove:

a. $\vdash \forall_w[\mathsf{Truth}@w = \mathsf{true}]$

b. $\vdash \forall_w[\mathsf{Falsity}@w = \mathsf{false}]$

c. $\vdash \forall_{w,p}[(\mathsf{not'}(p))@w = (\sim p)@w]$

d. $\vdash \forall_{w,p,q}[(p \text{ }\mathsf{and'}\text{ } q)@w = (p@w \wedge q@w)]$

e. $\vdash \forall_{w,p,q}[(p \text{ }\mathsf{or'}\text{ } q)@w = (p@w \vee q@w)]$

f. $\vdash \forall_{w,p,q}[(p \text{ }\mathsf{implies'}\text{ } q)@w = (p@w \supset q@w)]$

(10) **Entailment vs. Implication**

Utterances of the form 'S entails S' are analytic, not contingent. We can capture this with a meaning postulate using a constant $\mathsf{entails'}$: (Prop $\wedge$ Prop) $\supset$ Prop:

$$\vdash \forall_{w,p,q}[(p \text{ }\mathsf{entails'}\text{ } q)@w = \forall_{w'}(p@w' \supset q@w')]$$

This should be compared with (9f) directly above.

A HIGHER-ORDER
THEORY OF EXTENSIONS

(11) **What is the Extension of a Meaning at a World?**

a. Unlike standard PWS, we can't just evaluate the meaning at the world, because meanings are not intensions (functions from worlds to extensions), but rather *hyperintensions*.

b. Instead, we will define, for each hyperintensional type A, a function that assigns, to each hyperintension of type A, a function from worlds to things of type $\mathrm{Ext}(A)$.

c. That is, we have functions that tell what extensions hyperintensions have, at every world.

d. To talk about these functions within our higher-order theory, we introduce a family of constants (parametrized by $A \in \mathrm{HYPER}$)

$$\mathsf{ext}_A : A \supset (\mathrm{World} \supset \mathrm{Ext}(A))$$

Usually we omit the subscripts unless confusion could arise.

e. How should these functions be defined?

(12) **The Extension of a Proposition at a World**
Obviously, we want the extension of a proposition p at a world w to be true iff p is true at w! That is:

$\vdash \forall_{p,w}[\mathsf{ext}(p)(w) = p@w]$

(13) **What is the Extension of an Individual at a World?**

a. If the individual happens to be the meaning of a name, e.g. *Venus*, and if Kripke is right that names have rigid meanings, then we could just write a meaning postulate such as:

$\vdash \forall_w[\mathsf{ext}(\mathsf{Venus'})(w) = \mathsf{v}]$

where v is a constant for the planet Venus.

b. But what about a nonrigid individual i? For a given world w, what is the extension of i there?

c. It seems that the answer to that question should be one of the facts (true propositions) of w.

d. Now let's formalize that intuition.

(14) **The Extension of an Individual at a World**

a. We assume there is a function from individual-entity pairs to propositions, denoted by a constant $\mathsf{has\text{-}as\text{-}extension} : (\mathrm{Ind} \wedge \mathrm{Ent}) \supset \mathrm{Prop})$ subject to the following axiom:

$$\vdash \forall_{i,e,w}[(\mathsf{ext}(i)(w) = e) \leftrightarrow \mathsf{has\text{-}as\text{-}extension}(i, e)@w]$$

b. So for hyperintensions of both of the basic hyperintensional types (Prop and Ind), what the extension at any world w is dictated by the facts (true propositions) of w.

(15) **The Remaining Cases**

a. $\vdash \forall_{w,u}[\mathsf{ext}_{\mathrm{T}}(u)(w) = *]$

b. $\vdash \forall_{w,z}[(\mathsf{ext}_{A \wedge B}(z)(w) = (\mathsf{ext}_A(\pi(z))(w), \mathsf{ext}_B(\pi'(z))(w)$

c. $\vdash \forall_{w,f}(\mathsf{ext}_{A \supset B}(f)(w) = \lambda_{x \in A}\mathsf{ext}_B(f(x))(w))$

In other words:

a. A vacuous meaning has vacuous extension.

b. The extension of an ordered-pair meaning is the ordered pair of the extensions of the components.

c. The extension at w of a meaning that is a function is another function with the same domain: for each hyperintension in the domain, apply the meaning to it, and then take the extension at w of the resulting value.

(16) **An Example**

It is the last clause (15c) that is interesting.

a. For example, consider the individual property (type Ind $\supset$ Prop) of caninity.
b. The extension of this property at a world w is the set of all individuals i such that the proposition that i is a dog is true at w, i.e. the set of all w-dogs.
c. So the extension at each world is defined in terms of the meaning, not the other way around (as in standard PWS).
d. This makes intuitive sense: we figure out which things are dogs by looking at each thing and seeing if it is a dog.
e. Whereas on the standard approach, we figure out what *dog* means by, in every world, finding all the dogs there!

(17) **Is Reference Compositional?**

a. Frege said yes, and Montague organized his theory in such a way that his answer was also yes.

b. Frege even had to pay a price to make reference compositional: he had to stipulate that in certain contexts, the reference of an expression was its customary sense!

c. But from a linguistic point of view, there is no reason to think that reference (as opposed to meaning) is compositional.

d. The evidence is that we can figure out what utterances mean without having a clue what the contingent facts are.

e. Once we figure out what an utterance means, we can then figure out the reference if we know enough contingent facts, because reference is jointly determined by meaning together with contingent fact.

f. Our theory is consistent with this view, and we don't have to pay Frege's price.

(18) **Another Example**

a. For example, to decide whether it is true that Paris Hilton believes snow is white, we don't have to know anything about snow.

b. Instead, we have to know whether the proposition expressed by 'snow is white' is one of the ones that Hilton believes.

c. The moral is that, in order to account for the communicative function of language, we *do* have to assume *meaning* is compositional (otherwise how would we figure out what complex expressions mean?), but there does not seem to be any reason to assume *reference* is compositional.

WHEN ARE HYPERINTENSIONS EQUIVALENT?

(19) **Three Grades of Equality**

We introduce three different families of constants of type $(A \wedge A) \supset$ Prop (for $A \in$ HYPER):

a. equals_A is interpreted as the meaning of the verb *equals*. This has 'true equality' as its extension, as expressed in this meaning postulate:

$$\vdash \forall_{w,x,y}[(x\ \mathsf{equals}\ y)@w = (x = y)]$$

b. equiv_A is interpreted as the meaning of the term-of-art *is hyperintensionally equivalent to*, subject to the meaning postulate.

$$\vdash \forall_{w,x,y}[(x\ \mathsf{equiv}\ y)@w = \forall_{w'}(\mathsf{ext}(x)(w') = \mathsf{ext}(y)(w'))]$$

c. coext_A is interpreted as the meaning of the term-of-art *is coextensive with*, subject to the meaning postulate:

$$\vdash \forall_{w,x,y}[(x\ \mathsf{coext}\ y)@w = (\mathsf{ext}(x)(w) = \mathsf{ext}(y)(w))]$$

(20) **In Plain English** . . .

a. An utterance of the form ‘a equals b’ expresses that a and b are the same hyperintension.

b. An utterance of the form ‘a is hyperintensionally equivalent to b’ expresses that, at every world, a and b have the same extension at that world.

c. An utterance of the form ‘a is coextensive with b’ expresses that a and b have the same extension in the world that is actual relative to the utterance context.

(21) **Basic Facts of Hyperintensional Equivalence**

a. Hyperintensional equivalence is intermediate in strength between equality and coextensiveness:

   i. $\vdash \forall_{x,y}[(x \;\mathsf{equals}\; y) \models (x \;\mathsf{equiv}\; y)]$

   ii. $\vdash \forall_{x,y}[(x \;\mathsf{equiv}\; y) \models (x \;\mathsf{coext}\; y)]$

b. Hyperintensional equivalence is a generalization of truth-conditional equivalence, in the sense that two propositions are hyperintensionally equivalent iff they entail each other.

c. The preceding is just a restatement of Stone's Lemma.

(22) **Intensions Revisited**

a. Just because we use hyperintensions for meanings doesn't mean we don't have intensions (functions from worlds to extensions).

b. In fact, for any hyperintensional type A, ext_A is interpreted as a function from A-hyperintensions to functions from from worlds to things of type $\mathrm{Ext}(A)$:

   $\vdash \mathsf{ext}_A : A \supset (\mathrm{World} \supset \mathrm{Ext}(A))$

c. Examples:

   i. ext of an individual is a function from worlds to entities

   ii. ext of a property of individuals is a function from worlds to (characteristic functions of) sets of individuals

   iii. ext of a proposition is a function from worlds to truth values

d. In short, ext converts hyperintensions into their standard PWS counterparts!

(23) **Intensions and Hyperintensions Compared**

a. In hyperintensional semantics, we can *define* a type to be **intensional** if it is of the form World $\supset$ Ext(A) for some $A \in$ HYPER.

b. For two hyperintensions, being equivalent (equiv) just means corresponding to the same intension, e.g.

   i. *Hesperus* and *Phosphorus* (assuming rigidity of names)

   ii. *woodchuck* and *groundhog*

   iii. *Paris Hilton is Paris Hilton* and whichever is true, the Riemann Hypotheses or its denial.

(24) **Totally Stoned Out**

a. In particular, for $A = \text{Prop}$:

$\vdash \mathsf{ext}_{\text{Prop}} : \text{Prop} \supset (\text{World} \supset \text{Ext}(\text{Prop}))$

b. But

$\text{Ext}(\text{Prop}) =_{\text{def}} \text{Bool}$

and

$\text{World} =_{\text{def}} (\text{Prop} \supset \text{Bool})_{\mathsf{u}}$

c. And so:

$\vdash \mathsf{ext}_{\text{Prop}} : \text{Prop} \supset ((\text{Prop} \supset \text{Bool})_{\mathsf{u}} \supset \text{Bool})$

d. Specifically:
$\vdash \mathsf{ext}_{\text{Prop}} = \lambda_p \lambda_w (p@w)$

e. This is precisely the Stone mapping that maps each member of a boolean preorder to the set of ultrafilters to which it belongs.

(25) **Stalnakerian Hyperintensionality**

a. Suppose you like some aspects of hyperintensional semantics, but you believe Stalnaker's arguments that entailment is antisymmetric.

b. In that case, you can just take the theory we have so far and add the Antisymmetry axiom
$\vdash \forall_{p,q}[(p \equiv q) \supset (p = q)]$

c. Then (in an intepretation) the Stone mapping is an injection (as in Stone's original Representation Theorem for boolean algebras).

d. So you have something more like standard PWS semantics, except that the Nonprincipal Ultrafilters Problem and the Paris Hilton Omniscience Problem do not arise.

e. In this setup, the boolean algebra of propositions is isomorphic (via the Stone embedding) to a subalgebra of a powerset algebra (viz. the powerset of the set of ulktrafilters), without having to maintain (as standard PWS does) that it *is* a powerset algebra.

(26) **The Heart of the Problem**

The central foundational problem of standard PWS its failure to be informed by the Stone Representation Theorem.

a. Generalized to boolean preorders (i.e. not requiring antisymmetry), Stone gives a boolean homomorphism from propositions to sets of worlds (in the sense of maximal consistent sets of propositions, i.e. ultrafilters).

b. But there is no good reason to assume that this homomorphism is either injective or surjective; in fact either assumption leads to problems.

(27) **The Problem with Injectivity**

a. If the Stone mapping on propositions were injective, then entailment would be antisymmetric.

b. That leads to the Logical Omnisicence Problem, the propositional manifestation of the Granularity Problem.

(28) **One Problem with Surjectivity**

a. If the Stone mapping on propositions were surjective, then there would be propositions whose images are singleton sets (of ultrafilters).

b. These are the source of the Paris Hilton Omniscience Problem.

(29) **Another Problem with Surjectivity**

a. Since there must be an infinite number of equivalence classes of propositions, there must also (assuming Choice) be nonprincipal ultrafilters. Let u be one of them.

b. By surjectivity, there must be a proposition p whose image is $\{u\}$.

c. By definition of the Stone mapping, u is the *only* ultrafilter with p as a member.

d. Since u is not principal, it does not have a least element, so p is not least in u.

e. So there must be $q \in u$ such that q entails p but p does not entail q.

f. By Stone's Lemma, there must be an ultrafilter v such that $p \in v$ but $q \notin v$.

g. But since $p \in v$, $v = u$.

h. So $q \notin u$, a contradiction.

i. This is the Nonprincipal Ultrafilters Problem.

LECTURE EIGHT: QUANTIFIERS AND MODALITY

(30) **Goals of Lecture Eight**

- To make sense of the notion of **extensionality** in the hyperintensional setting
- To show that, in hyperintensional semantics, **quantificational meanings** work as expected
- To give hyperintensional analsyes of some basic modal concepts

GENERALIZED QUANTIFIERS IN HYPERINTENSIONAL SEMANTICS

(31) **A Preliminary Notion: Extensionality**

a. In standard PWS, a property is called **extensional** iff whether or not an intension has the property depends only on the intension's extension. Example:

b. Asssuming Zog is one of the Ancients, being seen by Zog is an extensional property. So if Zog sees Hesperus, then Zog must also see Phosphorus (since they have the same extension).

c. But being worshipped by Zog is not an extensional property: Zog might have worshipped Hesperus but not Phosphorus.

(32) **Extensionality in Hyperintensional Semantics**

a. In hyperintensional semantics, an A-**property** is something of type $A \supset$ Prop. E.g.

   i. groundhog' : Ind $\supset$ Prop is interpreted as an **individual property**

   ii. obvious : Prop $\supset$ Prop is intepreted as a **propositional property**.

b. If $A \in$ HYPER, we say an A-property f is **extensional** iff, at every world w, for any two A-hyperintensions a and b, if a and b are coextensive at w, then $f(a)$ and $f(b)$ have the same truth value.

c. More generally, we say a functional hyperintension $f : A \supset B$ ($A, B \in$ HYPER) is **extensional** iff, at every world w, for any two A-hyperintensions a and b, if a and b are coextensive at w, then so are $f(a)$ and $f(b)$.

(33) **Formalizing Extensionality in Hyperintensional Semantics**

a. We introduce the family of constants $\mathsf{extl}_{A,B} : (A \supset B) \supset \text{Bool}$ ($A, B \in$ HYPER), intepreted as the predicate (on hyperintensions of type $A \supset B$) of being extensional.

b. These are subject to the axioms:

$$\vdash \forall_{f \in A \supset B}[\mathsf{extl}(f) = \forall_{w,x,y}[(x \;\mathsf{coext}\; y)@w \supset (f(x) \;\mathsf{coext}\; f(y))@w]]$$

(34) **Examples**

a. Seen as properties of pairs of propositions, the meanings of the English logic words *and*, *or*, and *if . . . then* are extensional, e.g. the truth value of (p and' q) depends only on the extension of (p, q) (a pair of truth values).

b. Likewise, the meaning of *it is not the case that* is an extensional property of propositions.

c. These were already shown in (9) above.

d. In general, an extensional property of (tuples of) propositions is called a **truth-conditional propositional operator**.

(35) **Another Example**

a. Being a groundhog is an extensional property:

$\vdash \forall_{w,x,y}[(x \textsf{ coext } y)@w \supset (\textsf{groundhog'}(x)@w \leftrightarrow \textsf{groundhog'}(y)@w)]$

b. So (assuming the meaning of *Miss America* is a nonrigid individual), in a world where Miss America and Dick Cheney are coextensive, Dick Cheney is a groundhog in that world iff Miss America is a groundhog in that world.

c. The way we *say* in the theory that being a groundhog is an extensional property is with the meaning postulate:

$\vdash \textsf{extl}(\textsf{groundhog'})$

(36) **Definitions (Quantifiers and Determiners)**

For $A \in \text{HYPER}$,

a. an A-**quantifier** is an extensional function of type $(A \supset \text{Prop}) \supset \text{Prop}$, i.e. an extensional property of A-properties.

b. an A-**determiner** is an extensional function of type $((A \supset \text{Prop}) \wedge (A \supset \text{Prop})) \supset \text{Prop}$, i.e. an extensional property of pairs of A-properties.

(37) **Examples of Determiners**

a. We introduce families of constants (parametrized by $A \in$HYPER) interpreted as the meanings of the English determiners *every*, *some*, and *no* :

$$\vdash \mathsf{every'}_A, \mathsf{some'}_A, \mathsf{no'}_A : ((A \supset \text{Prop}) \wedge (A \supset \text{Prop})) \supset \text{Prop}$$

b. The expected truth-conditional behavior of these determiners is given by the following meaning postulates:

$$\vdash \forall_{w,P,Q}[\mathsf{every'}(P,Q)@w = \forall_x(P(x)@w \supset Q(x)@w)]$$

$$\vdash \forall_{w,P,Q}[\mathsf{some'}(P,Q)@w = \exists_x(P(x)@w \wedge Q(x)@w)]$$

$$\vdash \forall_{w,P,Q}[\mathsf{no'}(P,Q)@w = \sim\exists_x(P(x)@w \wedge Q(x)@w)]$$

A FIRST LOOK AT MODALITY IN HYPERINTENSIONAL SEMANTICS

(38) **Definition (Intensionality)**

a. We call a functional hyperintension **intensional** if its application to equivalent arguments yields equivalent values.

b. For properties of propositions (i.e. functions of type Prop $\supset$ Prop), intensionality corresponds to the traditional concept of **substitutivity** (preservation of truth value, at all worlds, upon substitution of the argument by something equivalent).

c. To assert intensionality within the theory, we introduce a family of constants $\mathsf{intl}_{A,B} : (A \supset B) \supset \text{Bool}$ ($A, B \in \text{HYPER}$), interpreted as the predicate (on functions of of type $A \supset B$) of being intensional.

d. These are subject to the axioms:

$$\vdash \forall_{f \in A \supset B}[\mathsf{intl}(f) = \forall_{x,y}[(x \equiv y) \supset (f(x) \equiv f(y))]]$$

e. It's easy to see that any functional hyperintension which is extensional is also intensional, but not conversely.

(39) **Basic Facts about Intensionality**

a. This concept has no interesting counterpart in standard PWS since there equivalence of intensions reduces to equality (and so *all* functions are trivially intensional).

b. So we can ask a question that is unaskable in standard PWS: are there any interesting intensional meanings (other than extensional ones)?

c. Any example of a *nonintensional* hyperintension is going to correspond to a problem for standard PWS, e.g.

   i. the individual property of being worshipped by Zog (cf. (31))

   ii. the property of propositions of being known by Paris Hilton

(40) **Example: Alethic Necessity**

a. We say a proposition is (**alethically**) **necessary** iff it is a top relative to entailment, or equivalently (by Stone's Lemma), iff it is true at all worlds.

b. We introduce the constant nec : Prop ⊃ Prop to be interpreted as althetic necessity.

c. This is subject to the axiom:

$\vdash \forall_p[\mathsf{nec}(p) \equiv (p \; \mathsf{equiv} \; \mathsf{Truth}))$

d. Or, equivalently:

$\vdash \forall_{w,p}[\mathsf{nec}(p)@w = \forall_{w'}(p@w')]$

e. Clearly alethic necessity is intensional but not extensional.

(41) **What is a Modal Operator?**

a. We might consider *defining* a **modal operator** to be an intensional property of propositions which is nonextensional (thus excluding not only nonintensional operators but also truth-conditional ones).

b. Of course, further conditions traditionally associated with $\Box$ or $\Diamond$ modalities could be imposed, e.g.

   i. being either nonincreasing ($\vdash m(p) \models p$) or nondecreasing ($\vdash p \models m(p)$)

   ii. being monotonic (if $\vdash p \models q$ then $\vdash m(p) \models m(q)$), from which being intensional follows

   iii. being subidempotent ($\vdash m(m(p)) \models m(p)$) or superidempotent ($\vdash m(p) \models m(m(p))$)

   iv. preserving conjunction ($\vdash m(p \ \mathsf{and'}\ q) \equiv (m(p)\ \mathsf{and'}\ m(q))$) up to equivalence, etc.

(42) **Accessibility Relations Recalled**

a. In standard PWS, (unary) modal operators are characterized by **accessibility** relations, i.e. binary relations on (primitive) worlds.

b. For R an accessibility relation, the corresponding necessity operator $\Box_R$ maps each "standard proposition' (i.e. each set of worlds) p to the set of all worlds w such that p is true at (i.e. has as a member) every world R-accessible from w:

$$\Box_R(p) =_{\text{def}} \{w \in W \mid \forall_{w' \in W}[R(w, w') \supset w \in p]\}$$

(43) **Accessibility in Hyperintensional Semantics**

a. As usual, we turn everything around.

b. We start with a modal operator $\Box$, i.e. an intensional function of type Prop $\supset$ Prop.

c. Then the corresponding accessibility relation $R(\Box)$ is the set of all pairs of ultrafilters $\langle w, w' \rangle$ such that every proposition that has property $\Box$ at w is true at w'.

d. To axiomatize this, we introduce a constant R of type ((Prop $\supset$ Prop) $\supset$ ((World $\wedge$ World) $\supset$ Bool)), which will be interpreted as the function that maps each modal operator to the accessibility relation it induces.

e. This is subject to the following axiom:

$$\vdash \forall_{\Box \in \text{Prop} \supset \text{Prop}}[\mathsf{R}(\Box) = \lambda_{w.w'} \forall_p (\Box(p)@w \supset p@w')]$$

f. Equivalent modal operators have the same accessibility relation.

(44) **Modal Bases Recalled**

a. For Kratzer, a **modal base** is a function B that assigns to each (primitive) world a set of (standard) propositions.

b. The corresponding accessibility relation is then defined such that w' is B-accessible from w iff it belongs to every proposition in $B(w)$.

(45) **Modal Bases, Hyperintensionally**

a. In hyperintensional semantics, again we go the other way.

b. If $\Box$ is a modal operator, then the associated modal base is $\mathsf{ext}(\Box)$, which is $\lambda_w \lambda_p[\Box(p)@w]$: World $\supset$ (Prop $\supset$ Bool),

c. Then the hyperintensional counterpart of (44b) is (defining $\subseteq$ as usual in HOL):

$$\vdash \forall_\Box[\mathsf{R}(\Box) = \lambda_{w,w'}[\mathsf{ext}(\Box)(w) \subseteq w']]$$

d. This is just a paraphrase of our previous axiom (43e).

e. Except for dropping Antisymmetry, and for formalizing it in HOL, this is all in line with Jónsson and Tarski 1951.

(46) **Example (Alethic Necessity)**

a. In an an interpretation, the modal base $\mathsf{ext}(\mathsf{nec})$ maps each world to the set of tops (analytic truths).

b. And so, for each pair of worlds $\langle w, w' \rangle$, w' is accessible from w iff w' has every top as a member.

c. But every top belong to every ultrafilter, and so every world is accessible from every world.

d. It is not hard to show that

$$\vdash \forall_{p,q}[(p \text{ } \mathsf{entails} \text{ } q) \equiv (\mathsf{nec}(p \text{ } \mathsf{implies} \text{ } q))]$$