

# **HYPERINTENSIONAL SEMANTICS**

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(1) **Basic Orientation of this Course**

- a. Possible-worlds semantics (PWS) is well-established as a theory of natural-language meaning; there is no good reason to stop doing it.
- b. But the standard implementation of PWS, which we will call **intensional** semantics, suffers from various foundational problems whose gravity ranges from the merely annoying to the downright catastrophic.
- c. We will examine a new implementation of PWS, called **hyperintensional** semantics, that solves many of these problems.
- d. In hyperintensional semantics, we still have possible worlds, and they can be used for much the same purposes as they were in intensional semantics.
- e. The technical prerequisites for hyperintensional semantics are no harder than for intensional semantics, though they are not exactly the same.
- f. We will assume familiarity with intensional semantics (along the lines of Montague 1970 or Gallin 1975), and introduce additional technicalia as needed.

# TENTATIVE COURSE OVERVIEW (DOUBTLESS OVERLY AMBITIOUS)

## Day One

Lecture 1: Introduction and Motivation

Lecture 2: Problems with Standard Possible-Worlds Semantics

## Day Two

Lecture 3: Soft Actualism Defined and Algebraicized

Lecture 4: The Positive Typed Lambda Calculus

## Day Three

Lecture 5: Higher Order Logic with Subtypes

Lecture 6: Hyperintensions and Entailment

## Day Four

Lecture 7: Worlds, Extensions, and Equivalence

Lecture 8: Quantifiers and Modality

## Day Five

Lecture 9: Questions

Lecture 10: Wrap-Up

**LECTURE ONE:  
INTRODUCTION AND  
MOTIVATION**

## (2) **Goals of Lecture One**

- a. Review how semantics fits into linguistic theory.
- b. Review the main features of PWS.
- c. Review the standard (i.e. intensional) implementation of PWS, identifying:
  - i. those aspects of standard PWS which we will retain, and
  - ii. those aspects of standard PWS which lead to problems.

**SEMANTICS AND  
LINGUISTIC THEORY**



### (3) Linguistic Background

- a. According to most linguistic theories, linguistic expressions are analyzed into two or more distinct **components** or **levels**, one of which is a meaning, or some kind of formal expression (such as a P&P LF or an HPSG CONTENT) that can be interpreted as a meaning.
- b. Linguistic theories differ with respect to how many non-meaning (e.g. prosodic, phonological, morphological, syntactic, etc.) components there are.
- c. They also differ with respect to whether the components are
  - i. **parallel** (e.g. HPSG, LFG, CG, Simpler Syntax), or
  - ii. **sequentially related**, aka **cascaded**, as in P&P, where the meaning of a sentence is derived from a purely syntactic representation of it.
- d. As far as possible, this course will remain neutral and speak simply of the **meaning** of a linguistic expression, disregarding how it is connected to syntax
- e. But at certain points this will be hard to do (for example, in our discussion of **semantic compositionality** in Lecture 2).

(4) **Full Disclosure**

Of course I am not really neutral! My biases are explained in ‘Non-local dependencies via variable contexts’ for the Workshop on New Directions in Type-Theoretic Grammar’, also in the course reader.

## (5) Abstracting away from Context

- a. Of course what an utterance of a linguistic expression expresses is inextricably wrapped up with the utterance context, e.g.
  - i. each utterance changes the context
  - ii. the interpretation of the utterance depends on how various parameters of the expression uttered are anchored in the context.
- b. In this course we consider only **literal interpretation**, also known as **preferred content**.
- c. Thus we ignore dynamic and parametric aspects of meaning, and simply assume the context is fixed.
- d. Correspondingly, we use the term ‘(linguistic) expression’ as shorthand for ‘contextualized utterance of an expression’, and likewise for terms (such as ‘declarative sentence’, ‘interrogative sentence’, ‘name’, etc.) for categories of linguistic expressions.
- e. Presumably, the modifications to PWS proposed here do not affect the available options for ‘going dynamic’.

**THE MAIN FEATURES  
OF POSSIBLE-WORLDS SEMANTICS**

(6) **The Extralinguistic Ontological Basis of PWS**

- a. There are different ways things might be, called **(possible) worlds**.
- b. One of the worlds, the **actual** world, is how things are.
- c. There are things called **senses**, that are independent of worlds.
- d. Every ordered pair of a sense and a world uniquely determines a thing called the **extension** of that sense at that world.
- e. A sense is called **rigid** if it has the same extension at every world.
- f. There are different sorts of senses, and different sorts of extensions.
- g. Two particular sorts of senses are called **propositions** and **individual concepts** (or just **individuals** for short).
- h. Two particular sorts of extensions are called **truth values** and **entities**.
- i. At any world, the extension of a proposition is a truth value.
- j. At any world, the extension of an individual is an entity.
- k. There are exactly two truth values, called **true** and **false**.

## (7) Entailment and Truth-Conditional Equivalence

- a. One proposition is said to **entail** another iff, at every world where its extension is **true**, the extension of the other is also **true**.
- b. It is easy to see that entailment is a **preorder**, i.e. a reflexive and transitive relation.
- c. Two propositions are said to be (**truth-conditionally**) **equivalent** iff they entail each other.
- d. Crucially, **nothing forces equivalent propositions to be the same**. Put another way, **there is no reason to suppose that entailment is an order (antisymmetric preorder)**.
- e. If for some reason you became committed to a semantic theory in which entailment **was** antisymmetric, you might try to persuade yourself that that was okay, or even (making a virtue of necessity) that it was a desirable state of affairs.
- f. If, on the hand, you were **not** committed to such a theory, the possibility of entailment being antisymmetric would probably never even cross your mind.
- g. Hyperintensional semantics will involve no such commitment.

(8) **Assumptions of Possible Worlds Semantics (PWS)**

- a. A natural language (here, English) specifies a set of **expressions**.
- b. Each expression has a **meaning**, also called its **semantics**, which is a sense. The expression is said to **express** its meaning.
- c. At any world, the extension of an expression's meaning is called the expression's **reference**, or **denotation**, at that world. The expression is said to **refer to**, or **denote**, its reference (at that world).
- d. There are different sorts of expressions, among them **declarative sentences** and **names**.
- e. Declarative sentences express propositions (and therefore denote truth values).
- f. Names express individuals (and therefore denote entities).

(9) **Rigid Meanings**

- a. Many philosophers and linguists assume (following Kripke) that meanings of names are rigid (i.e. that a name denotes the same entity at every world). Our semantic theory should be able to accommodate this assumption as an option.
- b. A declarative sentence is said to be **analytic** if it has a rigid meaning, and **contingent** otherwise.
- c. Obviously the only options for an analytic sentence are:
  - i. to denote **true** at every world—such sentences are said to be **analytically true**; or
  - ii. to denote **false** at every world—such sentences are said to be **analytically false**.



(10) **Following From**

- a. A declarative sentence is said to be **true** (respectively, **false**) at a world if it denotes **true** (respectively, **false**) at that world.
- b. If  $S$  and  $S'$  are declarative sentences,  $S'$  is said to **follow from**  $S$  if the proposition expressed by  $S$  entails the proposition expressed by  $S'$ .
- c. In other words:  $S'$  follows from  $S$  if it is true at every world where  $S$  is true.

**THE STANDARD (I.E. INTENSIONAL) IMPLEMENTATION  
OF POSSIBLE-WORLDS SEMANTICS:  
ASPECTS WE WILL RETAIN**

(11) **The Standard Implementation of PWS: Logic**

The semantic theory is asserted by nonlogical axioms (called **meaning postulates**) in a higher-order logic similar to Henkin's (1950) formulation of Church's (1940) simple theory of types. This includes:

- a. A typed lambda calculus with a basic type Bool (traditionally called  $t$ ) for truth values;
- b. terms of type Bool are called **formulas**;
- c. equality constants  $=_A$  at all types;
- d. the usual logical constants (connectives and quantifiers) of classical predicate logic are definable in terms of the  $=_A$  and  $\lambda$ ;
- e. the familiar lambda-calculus term equivalences (conversion) are formalized as object-language axioms about the  $=_{;A}$ ;
- f. Henkin's axiom of **Boolean Extensionality** (here  $x$  and  $y$  range over formulas):

$$\vdash \forall_{x,y}[(x \leftrightarrow y) \rightarrow (x = y)]$$

identifies bi-implication with boolean equality.

(12) **The Standard Implementation of PWS: Logic (Continued)**

The resulting logic is:

- a. classical
- b. two-valued
- c. higher-order (allows quantification over variables of all types)
- d. sound and complete for unrestricted Henkin models. (The Axiom of Boolean Extensionality is needed for this.)

(13) **The Standard Implementation of PWS: Types**

- a. There are types for different sorts of semantically relevant objects. These include:
  - i. World (traditionally,  $s$ ), for worlds (following Gallin, not Montague)
  - ii. Prop, for propositions
  - iii. Ind, for individuals
  - iv. the basic type Ent (traditionally,  $e$ ), for entities
  - v. the basic type Bool (traditionally,  $t$ ), for truth values, already supplied by the underlying logic
- b. Note that for the first three types listed above, we have not yet specified which ones are basic; intensional semantics and hyperintensional semantics make different choices here!
- c. By virtue of the typed lambda calculus underlying the logic, for any two types  $A$  and  $B$ , there is a corresponding function type  $A \supset B$  (traditionally written  $\langle A, B \rangle$ ).

(14) **The Standard Implementation of PWS: Meanings of ‘Logic Words’**

Besides the boolean connectives from the logic, there are also terms whose interpretations are the meanings of the English ‘logic words’:

- a. **and**’, of type  $\text{Prop} \supset (\text{Prop} \supset \text{Prop})$ , for the meaning of the sentence conjunction *and*
- b. **or**’, of type  $\text{Prop} \supset (\text{Prop} \supset \text{Prop})$ , for the meaning of the sentence conjunction *or*
- c. **implies**’, of type  $\text{Prop} \supset (\text{Prop} \supset \text{Prop})$ , for the meaning of the sentence conjunction *if ... then*
- d. **not**’, of type  $\text{Prop} \supset \text{Prop}$ , for the meaning of *it is not the case that*

Note that we have not yet specified whether or not these terms are basic (i.e. constants). In intensional semantics, they are not, but in hyperintensional semantics, they will be.

(15) **The Standard Implementation of PWS: Entailment**

- a. There is a term  $\models$  of type  $\text{Prop} \supset (\text{Prop} \supset \text{Bool})$  whose interpretation is the entailment relation between propositions.
- b. In hyperintensional semantics, this term will be basic, but in intensional semantics it is not.
- c. The meanings of the English logic words behave as expected with respect to entailment. For example, we can prove all the following theorems about **and'** (here  $p, q, r$  are variables over propositions):
  - i.  $\vdash \forall_{p,q}((p \text{ and}' q) \models p)$
  - ii.  $\vdash \forall_{p,q}((p \text{ and}' q) \models q)$
  - iii.  $\vdash \forall_{p,q,r}[(p \models q) \wedge (p \models r) \rightarrow (p \models (q \text{ and}' r))]$
- d. As a consequence, in a model, the set of propositions has a boolean structure with
  - i. **and'** interpreted as a meet (greatest lower bound) operation;
  - ii. **or'** interpreted as a join (least upper bound) operation; and
  - iii. **implies'** interpreted as a relative complement operation.

**THE STANDARD (I.E. INTENSIONAL) IMPLEMENTATION  
OF POSSIBLE-WORLDS SEMANTICS:  
ASPECTS WE WILL DISCARD**



(16) **Problematic Aspects of Standard PWS: Worlds**

- a. Standard PWS takes the type World ( $s$  in traditional notation) to be a **basic** type.
- b. This accords with Montague's decision to follow Kripke (1963) in treating **worlds as theoretical primitives**.
- c. But in hyperintensional semantics, following an older tradition, the type World will be **defined** (as the type of **maximal consistent sets of propositions**).

(17) **Problematic Aspects of Standard PWS: Senses**

- a. Standard PWS implements senses as **intensions**, i.e. functions whose domain is the set of worlds. In particular:
  - i. the type Prop for propositions is the functional type  $\text{World} \supset \text{Bool}$  ( $\langle s, t \rangle$  in traditional notation). Modulo the standard identification of sets with their characteristic functions, this means **propositions are implemented as sets of worlds**.
  - ii. the type Ind for individuals is the functional type  $\text{World} \supset \text{Ent}$  ( $\langle s, e \rangle$  in traditional notation).
- b. But in hyperintensional semantics, Prop and Ind will be **basic types**.

(18) **Problematic Aspects of Standard PWS: Entailment**

- a. In standard PWS semantics, the entailment relation between propositions is the interpretation of the term  $\lambda_p \lambda_q \forall_w (p(w) \supset q(w))$ .
- b. In a model, Prop is (essentially) the powerset of the set of worlds, and entailment is (essentially) the subset inclusion relation on it.
- c. Since subset inclusion is an order (and therefore antisymmetric), it follows that truth-conditionally equivalent (i.e. mutually entailing) propositions must be identical.
- d. But in hyperintensional semantics, there is a constant  $\models$  interpreted as entailment, and axiomatized so that entailment is merely a preorder on the set of propositions (not an order).

(19) **Problematic Aspects of Standard PWS: Logic Words**

In standard PWS:

- a. **and'** is  $\lambda_p \lambda_q \lambda_w (p(w) \wedge q(w))$ , so the meaning of *and* is (essentially) intersection (of sets of worlds)
- b. **or'** is  $\lambda_p \lambda_q \lambda_w (p(w) \vee q(w))$ , so the meaning of *or* is (essentially) union (of sets of worlds)
- c. **implies'** is  $\lambda_p \lambda_q \lambda_w (p(w) \supset q(w))$ , so the meaning of *if ... then* is (essentially) relative complement (of sets of worlds)
- d. But in hyperintensional semantics, **and'**, **or'**, and **implies'** are just constants, axiomatized so as to make the set of propositions into a boolean preorder (but not a powerset algebra, in fact not even a boolean algebra).

(20) **Looking Ahead**

- a. In Lecture Two, we will see how these aspects of standard PWS claimed to be problematic really do lead to serious trouble.
- b. Fortunately, these problematic aspects do not really model any empirically observed attributes of linguistic meanings; they are just artifacts of the standard (intensional) implementation.
- c. Hyperintensional semantics will be free of these troubling aspects, but there is no tradeoff involved; that is, there is no price to pay for switching over from intensional PWS to hyperintensional PWS.

**LECTURE TWO:  
PROBLEMS WITH STANDARD  
POSSIBLE-WORLDS SEMANTICS**

(21) **Goals of Lecture Two**

- a. Review several aspects of the notorious **Granularity Problem**.
- b. Explain the vexing, yet hitherto unremarked, **Problem of Nonprincipal Ultrafilters**.
- c. Consider a puzzle about **Singleton Propositions**.

**A REVIEW OF  
THE GRANULARITY PROBLEM**



(22) **The Granularity Problem**

For a wide range of entailment patterns, standard PWS does not allow finely-grained enough meaning distinctions to make predictions consistent with robust intuitions.

We will briefly consider three notorious manifestations of this problem:

- a. Hesperus and Phosphorus
- b. Woodchucks and Groundhogs
- c. So-called Logical Omniscience

(23) **Background on Semantic Compositionality**

- a. As background to the Granularity Problem, we need an informal familiarity with the idea of **semantic compositionality**.
- b. To formalize it, we'd have to choose a specific theory of the syntax-semantics interface (but we are trying to stay neutral about that).
- c. Informally, **semantic compositionality** means that the meaning of a phrase (i.e. nonlexical expression) can be determined from
  - i. the meanings of the phrase's ICs (immediate constituents), and
  - ii. the semantic effect of the grammar rule (aka construction) that licensed the phrase.
- d. But this notion of compositionality presupposes that each constituent of a sentence has a meaning, and therefore that the syntax-semantics interface has a parallel (rather than cascaded) architecture—see (3c-i) above.

(24) **Background on Substitutivity**

- a. A consequence of Semantic Compositionality is **Substitutivity**, i.e., that in a linguistic expression, if a constituent expression is replaced by a different expression with the same meaning, the meaning of the whole expression remains unchanged.
- b. Since meaning (together with how things are) determines reference, such replacements also leave the reference of the whole expression unchanged.
- c. A special case is that if the whole expression is a declarative sentence, then such replacements leave the truth value of the whole sentence unchanged.

(25) **Hesperus and Phosphorus**

- a. Suppose for the sake of argument that some community in antiquity, the Ancients, called the morning star *Phosphorus* and the evening star *Hesperus*, unaware that they were the same celestial body (viz. the planet Venus).
- b. Then it seems evident that of the two sentences
  - i. The ancients believed Hesperus was Hesperus.
  - ii. The ancients believed Hesperus was Phosphorus.the first is true and the second is false.
- c. The obvious analysis (roughly, Frege's) is to say the two names *Hesperus* and *Phosphorus* have different meanings, so Semantic Compositionality does not force the the replacement of *Hesperus* by *Phosphorus* to preserve the truth value.

(26) **The Hesperus-Phosphorus Puzzle**

- a. Now let's suppose Kripke is right that a name has a rigid meaning (and so has the same reference no matter how things are). This idea is not inconsistent with the analysis in (25b).
- b. But in standard PWS, since names express functions from worlds to entities, *Hesperus* and *Phosphorus* must have the same meaning, viz. the constant function that maps each world to the planet Venus; and so
- c. In standard PWS, the two sentences in (25b) must have the same truth value!

(27) **Woodchucks and Groundhogs**

- a. For the sake of argument, consider utterances of the following two sentences in which *Jim* refers to Jim Lambek, and *Phil* refers to a certain actually existing groundhog in Punkxatawney, Pennsylvania:
  - i. Jim thinks Phil is a groundhog.
  - ii. Jim thinks Phil is a woodchuck.
- b. Empirically, it seems clear that these need not have the same truth value. For example, Jim might know that Phil figures prominently in the Punkxatawney Groundhog Day Festival, but wrongly believe woodchucks to be a different kind of animal.
- c. There is no problem with substitutivity as long as we assume the two common nouns *groundhog* and *woodchuck* have different meanings.

(28) **The Woodchuck-Groundhog Puzzle**

- a. The two common nouns *woodchuck* and *groundhog* are about as good examples of synonymy as one could hope to find: they are two different names for the same kind of animal.
- b. The usual way of handling such cases of synonymy in PWS (no matter whether intensional or hyperintensional) is to write a meaning postulate asserting that, at each world, the meanings of the two words (the model-theoretic interpretations of the two constants *woodchuck'* and *groundhog'*) have the same extension.
- c. But in *standard* PWS, the meanings of common nouns are intensions, specifically functions that map each world to the meaning's extension at that world (which will be a set of individuals).
- d. In light of the meaning postulate in (28b), it follows that *woodchuck* and *groundhog* have the same meaning in standard PWS, since the two functions have the same value at every point of their shared domain.
- e. So by substitutivity, the two sentences in (27a) must have the same truth value in standard PWS!

(29) **Paris Hilton and the Riemann Hypothesis (Prelude)**

- a. The Riemann Hypothesis is a certain (English) sentence  $R$  that asserts something about a certain function on the complex numbers called  $\zeta$  (the Riemann zeta-function).
- b. Just exactly which sentence  $R$  is is not relevant to our purposes.
- c. Louis de Branges (Purdue University) thinks he has proved  $R$ , but the referees cannot understand his proof. So the consensus in the mathematical community is that  $R$  remains an open conjecture.
- d. In short, nobody knows whether  $R$  (except maybe de Branges, but he has not persuaded anybody else that he does).
- e. So we can assert with confidence:  
Paris Hilton does not know whether  $R$ .



(30) **Paris and Riemann (Development)**

- a. Now consider the following two sentences:
  - i. Paris Hilton is Paris Hilton.
  - ii.  $S$ , where  $S$  is either (i)  $R$ , or (ii) *It is not the case that*  $R$ , whichever is true.
- b. Under widely accepted assumptions about assertions of identity and about mathematical truths, both (30a-i) and (30a-ii) are analytically true, and so they both denote **true** at every world.
- c. But according to standard PWS, declarative sentence meanings are functions that map each world to the truth value at that world of the sentence in question.
- d. And so the the two sentences (30a-i) and (30a-ii) have the same meaning, viz. the function that maps every world to **true**.
- e. Possibly being excessively charitable, let us assume that:  
Paris Hilton knows that Paris Hilton is Paris Hilton.
- f. Then it follows, by substitutivity, that:  
Paris Hilton knows that  $S$ .

(31) **Paris and Riemann (Finale)**

- a. Under widely accepted assumptions about indirect polar questions, one knows whether  $R$  iff either one knows that  $R$  (if  $R$  is true) or one knows that it is not the case that  $R$  (if  $R$  is false).
- b. It follows that one knows whether  $R$  iff one knows that  $S$ .
- c. And so, inevitably, according to standard PWS:  
Paris Hilton knows whether  $R$ .
- d. To keep things in perspective, please keep in mind that:
  - i.  $R$  (the Riemann Hypothesis) is the most celebrated open conjecture in all of mathematics, and
  - ii. Paris Hilton is a celebrity heiress devoted to parties and shopping.

(32) **The Logical Omniscience Problem**

The foregoing is an example of the so-called **Logical Omniscience Problem**: according to standard PWS, if one knows some analytic truth or other, than one knows *every* analytic truth.

**THE PROBLEM OF  
NONPRINCIPAL ULTRAFILTERS**

(33) **Ultrafilters in Power Sets**

Let  $S$  be any set,  $A = \wp(S)$  its power set, and  $U \subseteq A$  (i.e.  $U$  is a set of subsets of  $S$ ). Then  $U$  is called an **ultrafilter** of  $A$  iff the following three conditions hold:

- a. The intersection of any two members of  $U$  is also in  $U$ .
- b. For  $X$  and  $Y$  any two members of  $A$ , if  $X \in U$  and  $X \subseteq Y$ , then  $Y \in U$ .
- c. For every  $X \in A$ , either  $X \in U$  or  $S \setminus X \in U$ , but not both.

Note 1: It follows from the definitions that  $S \in U$ , but  $\emptyset \notin U$ .

Note 2: Later we will extend the concept of ultrafilter to a much more general setting.

(34) **Review of the Standard PWS Modelling**

- a. There is a set  $W$  of worlds.
- b. The set  $P$  of propositions is  $\wp(W)$ , the power set of  $W$ .
- c. Proposition  $p$  being true at world  $w$  is modelled by  $w$  being a set-theoretic member of  $p$ .
- d. Entailment is the subset inclusion relation on  $P$ .
- e. Propositional conjunction (the meaning of *and*) is intersection in  $P$ .
- f. Propositional disjunction (the meaning of *or*) is union in  $P$ .
- g. Propositional implication (the meaning of *if ... then*) is the relative complement operation in  $P$ .
- h. Propositional negation (the meaning of *it is not the case that*) is the complement operation in  $P$ .
- i.  $W$  (as a member of  $P$ ), the set of all worlds, is the meaning of every analytically true sentence.
- j.  $\emptyset$  (as a member of  $P$ ), the empty set of worlds, is the meaning of every analytically false sentence.

(35) **Maximal Consistent Sets of Propositions**

Still in the standard PWS setting, a set  $M$  of propositions is called a **maximal consistent set of propositions** iff it satisfies the following three conditions:

- a.  $M$  is **closed under conjunction**, i.e. for any two propositions  $p$  and  $q$ , if  $p$  and  $q$  are both in  $M$ , then so is their conjunction  $p$  and'  $q$ .
- b.  $M$  is **closed under entailment**, i.e. for any two propositions  $p$  and  $q$ , if  $p \in M$  and  $p$  entails  $q$ , then  $q \in M$ .
- c.  $M$  'resolves all issues', i.e. for every proposition  $p$ , either  $p$  or its negation **not**'  $p$  is in  $M$ , but not both.

It follows directly from the definitions that a maximal consistent set of propositions is the same thing as an ultrafilter of  $P = \wp(W)$ .

(36) **From Worlds to Maximal Consistent Sets**

- a. Still in the standard PWS setting, let  $m$  be the function from worlds to sets of propositions that maps each world  $w$  to the set of all propositions true at  $w$ , i.e.  $m : W \rightarrow \wp(P)$  such that for each  $w \in W$ ,  $m(w) = \{p \in P \mid w \in P\}$ .
- b. Then it is easy to verify that, for each world  $w$ ,  $m(w)$  is a maximal consistent set of propositions.
- c. Intuitively, it seems that a maximal consistent set of propositions can be thought of as completely specifying a way things might be.
- d. But *worlds* are supposed to be modelling ways things might be.
- e. Which is the better candidate for modelling ways things might be, worlds or maximal consistent sets of propositions?
- f. It would be nice if  $m$  were a surjection onto the set of maximal consistent sets of propositions (it is obviously an injection); then the preceding question would be a nonissue.
- g. But is  $m$  a surjection? That is, if  $U$  is a maximal consistent set of propositions, is there a world  $w$  such that  $U$  is the set of all propositions true at  $w$ ?



(37) **Principal and Nonprincipal Ultrafilters**

Let  $S$  be any set and  $A = \wp(S)$  its power set. Then:

- a. For each  $x \in S$ , the set  $U_x =_{\text{def}} \{X \in A \mid x \in X\}$  can be shown to be an ultrafilter. Such an ultrafilter is called a **principal** ultrafilter, and  $U_x$  is called the principal ultrafilter **generated** by  $x$ .
- b. If  $S$  is finite, the function from  $S$  to ultrafilters that maps each  $x \in S$  to  $U_x$  is bijective onto the set of ultrafilters, i.e. every ultrafilter is principal.
- c. But in ZFC (Zermelo-Fraenkel set theory plus the Axiom of Choice), it can be proved that if  $S$  is infinite, then not every ultrafilter is principal.

(38) **The Infinitude of the Set of Propositions**

- a. It is easy to show that there is a countable infinitude of English sentences no two of which follow from each other.
- b. One example:
  - i. Frege had exactly one cat.
  - ii. Frege had exactly two cats.
  - iii. Frege had exactly three cats. (Etc.)
- c. Another example:
  - i. Frege erred.
  - ii. Russell knew Frege erred.
  - iii. Frege knew Russell knew Frege erred, (Etc.)
  - iv. etc.
- d. So semantic theory must admit an infinitude of propositions.
- e. In standard PWS, the set  $W$  of worlds must be infinite; otherwise, since  $P = \wp(W)$ , there would be only finitely many propositions.

(39) **Nonprincipal Ultrafilters in Standard PWS**

- a. Applying (37c) to the case  $S = W, A = P$  shows that the function  $m$  from worlds to maximal consistent sets of propositions is not surjective.
- b. So there is a maximal consistent set of propositions  $U$  such that there is no world  $w$  whose set of true propositions is  $U$ .
- c. What should an advocate of standard PWS say about  $U$ ?
  - i. One option: deny the Axiom of Choice. This seems a high price to pay. And anyway, why should semantic theory get to dictate what ambient set theory we use?
  - ii. Admit that there are maximal consistent sets of propositions that don't correspond to any world, but argue somehow that they shouldn't really count as ways things might be. But how?
- d. Life would be a lot simpler if propositions (rather than worlds) were the theoretical primitives, and maximal consistent sets were the only way we had of modelling ways things might be.
- e. This is not a new idea; it predates standard PWS by a few decades.

**A PUZZLE ABOUT  
SINGLETON PROPOSITIONS**

(40) **Singleton Propositions**

- a. Still in the context of standard PWS, for any world  $w$ , we write  $p_w$  for  $\{w\}$ , the proposition that is true at  $w$  but not at any other world.
- b. Since there is an infinitude of worlds, in particular there is more than one, and so  $p_w$  is contingent.
- c. Note that  $p_w$  is the conjunction of all propositions true at  $w$ .

(41) **Some Assumptions about Knowledge**

- a. For any individual  $i$  and any proposition  $p$ , we will abbreviate by  $k_{i,p}$  the proposition that  $i$  knows that  $p$ .
- b. Extensionally, knowledge is a binary relation between individuals and propositions.
- c. Knowledge is **factive**: if  $k_{i,p}$  is true at  $w$ , then so is  $p$ .
- d. No matter what one thinks about *logical* omniscience, it seems clear that for a given *contingent* proposition  $p$  and a given individual  $i$ , the proposition  $k_{i,p}$  is also contingent.

(42) **The Omniscience of Paris Hilton**

- a. But now let  $i$  be Paris Hilton,  $w_0$  be the actual world,  $p$  be  $p_{w_0}$ , and  $q$  be  $k_{i,p}$ .
- b. That is,  $q$  is the proposition that Paris Hilton knows the conjunction of every actually true proposition (not merely the analytic truths).
- c. By factivity of knowing,  $q$  must be false at every nonactual world.
- d. But  $q$  is contingent, so must be true in the actual world.
- e. So evidently Paris Hilton is omniscient *simpliciter*!
- f. This goes way beyond Paris Hilton being merely *logically* omniscient (e.g. knowing whether the Riemann hypothesis is true), which (assuming standard PWS) we have already had to accept.
- g. For example, one of the conjuncts of  $p_{w_0}$  is the one expressed by an English sentence that correctly asserts the number of molybdenum atoms in the Crab Nebula.
- h. Doubtless, standard PWS fans can wiggle out of this somehow.
- i. But why not instead just abandon the modelling of propositions as sets of worlds, so that the problem does not even arise?