Foundations of Language Science and Technology

Predicate Logic

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Textbooks

L.T.F. Gamut. Logic, Language and Meaning. Volume I: Introduction to Logic, University of Chicago Press, 1991.

Barbara H. Partee, Alice ter Meulen, Robert E. Wall. Mathematical Methods in Linguistics. Springer, 1990.

Arguments

(1) a. If it rains, then the street is wet
b. It rains
c. Therefore, the street is wet

(2) a. If it rains, then the street is wet
b. The street is not wet
c. Therefore, it does not rain

(3) a. If it rains, then the street is wet
b. The street is wet
c. Therefore, it rains

Arguments

(4) a. All man are mortal
b. Sokrates is a man
c. Therefore, Sokrates is mortal

$\forall x(H(x) \rightarrow M(x))$
$H(s)$

$\therefore M(s)$

Predicate Logic – Vocabulary

- **Non-logical expressions:**
  - Set of individual constants: CON (possibly empty)
  - Set of n-place relation constants: $PRED^n$, for all $n \geq 0$ (possibly empty)
- **Infinite set of individual variables:** VAR (infinite set)

Predicate Logic – Syntax

- **Terms:** TERM = VAR ∪ CON
- **Atomic formulas:**
  - $R(t_1,\dots, t_n)$ for $R \in PRED^n$ and $t_1, \dots, t_n \in$ TERM
  - $t_1 = t_2$ for $t_1, t_2 \in$ TERM
- **Well-formed formulas (WFF)**
  - all atomic formulas are WFF
  - if φ and ψ are WFF, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \leftrightarrow \psi)$ are WFF
  - if $x \in$ VAR, and φ is a WFF, then $\forall x\varphi$ and $\exists x\varphi$ are WFF
  - nothing else is a WFF

Quantification

∃xA – "there is an x such that A"

∀xA – "for every x it is the case that A"

Exercise – Formalization

(1) *John and Mary work*

↦ work'(j) ∧ work'(m)

(2) *A student works*

↦ ∃x(student'(x) ∧ work'(x))

(3) *A blond student works*

↦ ∃x(student'(x) ∧ blond'(x) ∧ work'(x))

(4) *A blond student works hard*

↦ ∃x(student'(x) ∧ blond'(x) ∧ work-hard'(x))

Exercise – Translate into PL

(1) *Mary loves a student*

↦ ∃x(student'(x) ∧ love'(m, x))

(2) *Every student works*

↦ ∀x (student'(x) → work'(x))

(3) *Nobody flunked*

↦ ¬∃x flunk'(x)

(4) *Barking dogs don't bite*

↦ ∀x ((dog'(x) ∧ bark'(x)) → ¬bite'(x))

Scope

- If ∀xφ (∃xφ) is a subformula of a formula ψ, then φ is the **scope** of this occurrence of ∀x (∃x) in ψ.
- We distinguish distinct occurrences of quantifiers as there are formulae like ∀xA(x) ∧ ∀xB(x).
- Examples:
  - ∃x (∀y (T(y) ↔ x=y) ∧ F(x))
  - ∀x A(x) ∧ ∀x B(x)

Free and Bound Variables

- An occurrence of a variable x in a formula φ is **free in** φ if this occurrence of x does not fall within the scope of a quantifier ∀x or ∃x in φ.
- If ∀xψ (∃xψ) is a subformula of φ and x is free in ψ, then this occurrence of x is **bound by** this occurrence of the quantifier ∀x (∃x).
- Examples:
  - ∀x(A(x) ∧ B(x)) – x occurs bound in B(x)
  - ∀x A(x) ∧ B(x) – x occurs free in B(x)
- **A sentence** is a formula without free variables.

Predicate Logic – Semantics

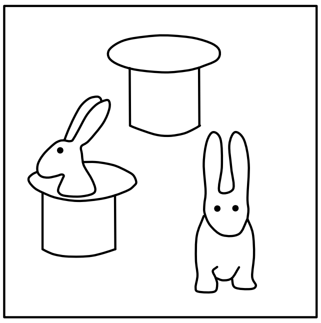
- Expressions of Predicate Logic are interpreted relative to **model structures** and **variable assignments**.
- Model structures are our "mathematical picture" of the world. They provide interpretations for the non-logical symbols (predicate symbols, individual constants).
- Variable assignments provide interpretations for variables.

Model structures

- **Model structure:** $M = \langle U_M, V_M \rangle$
  - U_M is non-empty set – the "universe"
  - V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:
    - $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
    - $V_M(c) \in U_M$ if c is an individual constant
- **Assignment function** for variables $g: VAR \rightarrow U_M$

Model structures – Example

$M = \langle U_M, V_M \rangle$
$U_M = \{ r_1, r_2, h_1, h_2 \}$
$V_M(\text{vincent}) = r_1$
$V_M(\text{mia}) = r_2$
$V_M(\text{rabbit}) = \{ r_1, r_2 \}$
$V_M(\text{white}) = \{ r_2 \}$
$V_M(\text{hat}) = \{ h_1, h_2 \}$
$V_M(\text{in}) = \{ (r_1, h_1) \}$



Interpretation (terms)

Interpretation of terms with respect to a model structure M and a variable assignment g:

$$[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$$

Interpretation (atomic formulas)

Interpretation of (atomic) formulas with respect to a model structure M and variable assignment g:

$[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$

$[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$

Is Vincent a rabbit?

$[\![\text{rabbit(vincent)}]\!]^{M,g} = 1$

- iff $[\![\text{vincent}]\!]^{M,g} \in V_M(\text{rabbit})$
- iff $V_M(\text{vincent}) \in V_M(\text{rabbit})$

$$
\begin{aligned}
M &= (U_M, V_M) \\
U_M &= \{ r_1, r_2, h_1, h_2 \} \\
V_M(\text{vincent}) &= r_1 \\
V_M(\text{mia}) &= r_2 \\
V_M(\text{rabbit}) &= \{ r_1, r_2 \} \\
V_M(\text{white}) &= \{ r_2 \} \\
V_M(\text{hat}) &= \{ h_1, h_2 \} \\
V_M(\text{in}) &= \{ (r_1, h_1) \}
\end{aligned}
$$

Interpretation (connectives)

Connectives are **truth-functional:** the truth-value of a complex expession is determined by the truth-values of their subformulas.

$[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

Truth-functional connectives

- A connective is **truth-functional** iff the truth value of any compound statement obtained by applying that connective is a function of the individual truth values of the constituent statements that form the compound.
- **Truth-functional connectives:**
  substituting sub-expressions with the same truth-value does not change the truth of the complete expression.

Truth-functional connectives

(1) *John bumped his head* ***and*** *he [John] is crying*

(2) *John bumped his head* ***and*** *it is raining*

(3) *John is crying*

(4) *It is raining*

- Assume that (3) and (4) have the same truth-value.
  - Then (1) and (2) must have the same truth-value
  - ***and*** is a truth-functional connective

Truth-functional connectives

(1) *John is crying* ***because*** *he [John] bumped his head*

(2) *John is crying* ***because*** *it is raining*

(3) *John bumped his head*

(4) *It is raining*

- Assume that (3) and (4) have the same truth-value.
  - (1) and (2) can have different truth-values
  - ⇒ ***because*** is not truth-functional

Is Vincent a white rabbit?

$[\![\text{rabbit(vincent)} \wedge \text{white(vincent)}]\!]^{M,g} = 1$

- iff $[\![\text{rabbit(vincent)}]\!]^{M,g} = 1$
  and $[\![\text{white(vincent)}]\!]^{M,g} = 1$
- iff $V_M(\text{vincent}) \in V_M(\text{rabbit})$
  and $V_M(\text{vincent}) \in V_M(\text{white})$

$$
\begin{aligned}
M &= (U_M, V_M) \\
U_M &= \{ r_1, r_2, h_1, h_2 \} \\
V_M(\text{vincent}) &= r_1 \\
V_M(\text{mia}) &= r_2 \\
V_M(\text{rabbit}) &= \{ r_1, r_2 \} \\
V_M(\text{white}) &= \{ r_2 \} \\
V_M(\text{hat}) &= \{ h_1, h_2 \} \\
V_M(\text{in}) &= \{ (r_1, h_1) \}
\end{aligned}
$$

Interpretation (quantifiers)

We want:

- $[\![\forall x A(x)]\!]^{M,g} = 1$ iff for every $d \in U_M$, $d \in [\![A]\!]^{M,g}$
- $[\![\exists x A(x)]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $d \in [\![A]\!]^{M,g}$

Interpretation (quantifiers)

- **Interpretation of formulas** with respect to a model structure M and variable assignment g:
  - $[\![\exists x \varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$
  - $[\![\forall x \varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$
- **g[x/d]** is the variable assignment which is identical to g except that it assigns the individual d to variable x.

$$
g[x/d](y) = \begin{cases} d & \text{if } x = y \\ g[x/d](y) = g(y) & \text{if } x \neq y \end{cases}
$$

Variable assignments

$$g[x/d](y) = \begin{cases} d & \text{if } x = y \\ g[x/d](y) = g(y) & \text{if } x \neq y \end{cases}$$

	x	y	z	u	…
g	a	b	c	d	…
g[x/a]	a	b	c	d	…
g[y/a]	a	a	c	d	…
g[y/g(z)]	a	c	c	d	…
g[y/a][u/a]	a	a	c	a	…
g[y/a][y/b]	a	b	c	d	…

Predicate Logic: Semantics

Interpretation of formulas with respect to a model structure M and variable assignment g:

$[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$

$[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$

$[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

$[\![\exists x\varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$

$[\![\forall x\varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$

Every rabbit is in a hat

$[\![\forall x(\text{rabbit}(x) \rightarrow \exists y(\text{hat}(y) \wedge \text{in}(x, y))]\!]^{M,\, g} = 1$

- iff … [⇒ whiteboard]

M	$= (U_M, V_M)$
U_M	$= \{ r_1, r_2, h_1, h_2 \}$
V_M(vincent)	$= r_1$
V_M(mia)	$= r_2$
V_M(rabbit)	$= \{ r_1, r_2 \}$
V_M(white)	$= \{ r_2 \}$
V_M(hat)	$= \{ h_1, h_2 \}$
V_M(in)	$= \{ (r_1, h_1) \}$

Not every rabbit is white

$[\![\neg\forall x(\text{rabbit}(x) \rightarrow \text{white}(x))]\!]^{M,g} = 1$

- iff … [⇒ whiteboard]

M	$= (U_M, V_M)$
U_M	$= \{ r_1, r_2, h_1, h_2 \}$
$V_M(\text{vincent})$	$= r_1$
$V_M(\text{mia})$	$= r_2$
$V_M(\text{rabbit})$	$= \{ r_1, r_2 \}$
$V_M(\text{white})$	$= \{ r_2 \}$
$V_M(\text{hat})$	$= \{ h_1, h_2 \}$
$V_M(\text{in})$	$= \{ (r_1, h_1) \}$

More Examples

- $[\![\exists x(\forall x\ A(x) \wedge B(x))]\!]^{M,g} = 1$ iff …
- $[\![\forall x\ A(x) \wedge B(x)]\!]^{M,g} = 1$ iff …
- $[\![\exists x\ \forall y\ L(x, y)]\!]^{M,g} = 1$ iff …
- $[\![\forall y\ \exists x\ L(x, y)]\!]^{M,g} = 1$ iff …

True, Valid, Satisfiable

- **A formula φ is true in a model structure M** iff $[\![\varphi]\!]^{M,g} = 1$ for every variable assignment g
- **A formula φ is valid (⊨ φ)** iff φ is true in all model structures
- **A formula φ is satisfiable** iff there is at least one model structure M such that φ is true in M

Satisfiable? Valid?

(1) $\forall x F(x) \rightarrow \exists x F(x)$

(2) $\exists x \forall y \Phi \rightarrow \forall y \exists x \Phi$

(3) $\exists x(F(x) \wedge \neg F(x))$

(4) $\exists x F(x) \vee \neg F(x)$

Entailment

- **A set of formulas Γ is (simultaneously) satisfiable** iff there is a model structure M such that every formula in Γ is true in M ("M satisfies Γ," or "M is a model of Γ")
- **Γ is contradictory** if Γ is not satisfiable.
- **Γ entails a formula φ (Γ ⊨ φ)** iff φ is true in every model structure that satisfies Γ

Example [⇒ Blackboard]

(1) *Not every blond student passed*

(2) *Not every student passed*

Some logical laws

Quantifier negation

- $\neg\forall x\varphi \Leftrightarrow \exists x\neg\varphi$

Quantifier distribution

- $\forall x(\varphi \wedge \Psi) \Leftrightarrow \forall x\varphi \wedge \forall x\Psi$
- $\exists x(\varphi \vee \Psi) \Leftrightarrow \exists x\varphi \vee \exists x\Psi$

Quantifier (in-)dependence

- $\forall x\forall y\varphi \Leftrightarrow \forall y\forall x\varphi$
- $\exists x\exists y\varphi \Leftrightarrow \exists y\exists x\varphi$
- $\exists x\forall y\varphi \Rightarrow \forall y\exists x\varphi$ (but not vice versa)

Some logical laws

Quantifier movement

- $\varphi \rightarrow \forall x\Psi \quad \Leftrightarrow \quad \forall x(\varphi \rightarrow \Psi)$
- $\varphi \rightarrow \exists x\Psi \quad \Leftrightarrow \quad \exists x(\varphi \rightarrow \Psi)$
- $\forall x\Psi \rightarrow \varphi \quad \Leftrightarrow \quad \forall x(\Psi \rightarrow \varphi)$
- $\exists x\Psi \rightarrow \varphi \quad \Leftrightarrow \quad \exists x(\Psi \rightarrow \varphi)$

… provided that x does not occur free in φ