

Semantic Theory

Week 1 – Predicate Logic

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Information about this course

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Recommended literature:

- Gamut: Logic, Language, and Meaning, Vol. 2, University of Chicago Press, 1991
- Kamp and Reyle: From Discourse to Logic, Kluwer, 1993
- Winter: Elements of Formal Semantics, Edinburgh University Press, 2016
(first three chapters freely available for download: <http://www.phil.uu.nl/~yoad/efs/main.html>)

Final exam:

- Exam date (provisional): 18.07.2019

Part I: Sentence semantics



Sentence meaning

- Tina is tall and thin \Rightarrow Tina is tall
- Tina is tall, and Ms. Turner is not tall \Rightarrow Tina is not Ms. Turner
- A dog entered the room \Rightarrow An animal entered the room
- Tweety is a bird \nRightarrow Tweety can fly

Entailment

Given an **indefeasible relation** between two natural language sentences S_1 and S_2 , where speakers intuitively judge S_2 to be true whenever S_1 is true, we say that S_1 **entails** S_2 , and denote it $S_1 \Rightarrow S_2$

Formalizing sentence meaning

How to **formally** describe the meaning of sentences?

- Defining differences between various linguistic *forms*
- Using *formal* mathematical methods

Truth-conditional semantics:

To know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true:

Sentence meaning = truth-conditions

Indirect interpretation

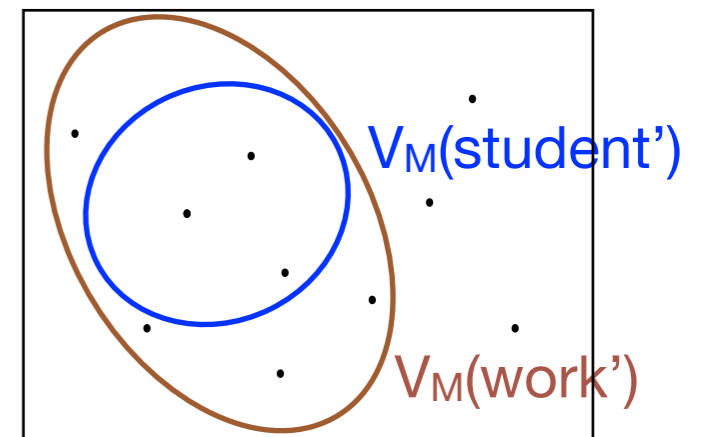
1. Translate sentences into logical formulas:

Every student works $\mapsto \forall x(\text{student}'(x) \rightarrow \text{work}'(x))$

2. Interpret these formulas in a logical model:

$\llbracket \forall x(\text{student}'(x) \rightarrow \text{work}'(x)) \rrbracket^{M,g} = 1$ iff

$V_M(\text{student}') \subseteq V_M(\text{work}')$



Step 1: Translation

Limits of propositional logic: propositions with internal structure

Every man is mortal.

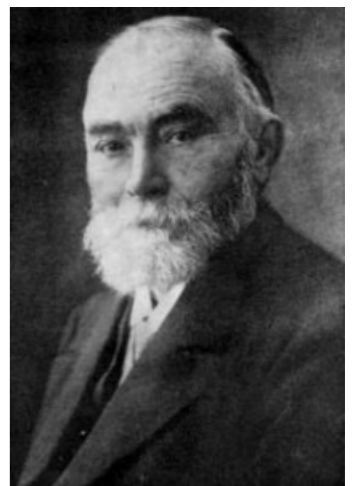
Socrates is a man.

Therefore, Socrates is mortal.

Solution: first-order predicate logic

predicates are expressions
that contain *arguments*
(constants and variables)

predication & quantification
over *individuals*



Gottlob Frege
Begriffsschrift (1879)

Predicate Logic: Vocabulary

Non-logical expressions:

Individual constants: CON

n-place relation constants: PRED^n , for all $n \geq 0$

Infinite set of individual variables: VAR

Logical connectives: $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \forall, \exists$

Brackets: (,)

Predicate Logic: Syntax

Terms: $\text{TERM} = \text{VAR} \cup \text{CON}$

Atomic formulas:

- $R(t_1, \dots, t_n)$ for $R \in \text{PRED}^n$ and $t_1, \dots, t_n \in \text{TERM}$
- $t_1 = t_2$ for $t_1, t_2 \in \text{TERM}$

Well-formed formula (WFF):

1. All atomic formulas are WFFs;
2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;
3. If $x \in \text{VAR}$, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFFs;
4. Nothing else is a WFF.

Variable binding

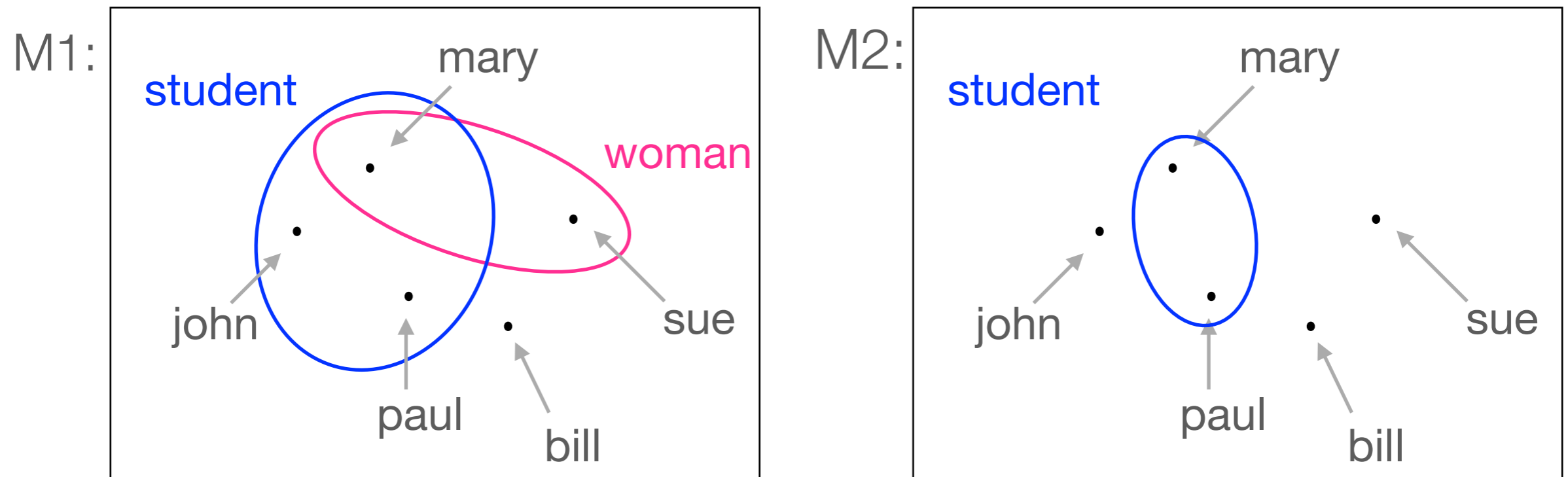
- Given a quantified formula $\forall x\phi$ (or $\exists x\phi$), we say that ϕ (and every part of ϕ) is in the **scope** of the quantifier $\forall x$ (or $\exists x$);
- A variable x is **bound** in formula ψ if x occurs in the scope of $\forall x$ or $\exists x$ in ψ ;
- If a variable is not bound in formula ψ , it occurs **free** in ψ ;
- A **closed formula** is a formula without free variables.

Formalizing Natural Language

1. *Bill loves Mary.*
2. *Bill reads an interesting book.*
3. *Every student reads a book.*
4. *Bill passed every exam.*
5. *Not every student answered every question.*
6. *Only Mary answered every question.*
7. *Mary is annoyed when someone is noisy.*
8. *Although nobody makes noise, Mary is annoyed.*

Step 2: Interpretation

Logical models are simplified representations of the state of affairs in the world



John is a student : for any M, $\llbracket \text{student}'(\text{john}) \rrbracket^M = 1$ iff $V_M(\text{john}) \in V_M(\text{student}')$

$V_{M1}(\text{john}) \in V_{M1}(\text{student}')$ therefore: $\llbracket \text{student}'(\text{john}) \rrbracket^{M1} = 1$

$V_{M2}(\text{john}) \notin V_{M2}(\text{student}')$ therefore: $\llbracket \text{student}'(\text{john}) \rrbracket^{M2} = 0$

A formal description of a model

Model $M = \langle U_M, V_M \rangle$, with:

- U_M is the universe of M and
- V_M is an interpretation function

$U_M = \{e_1, e_2, e_3, e_4, e_5\}$ **universe**

$V_M(\text{john}) = e_1$

...

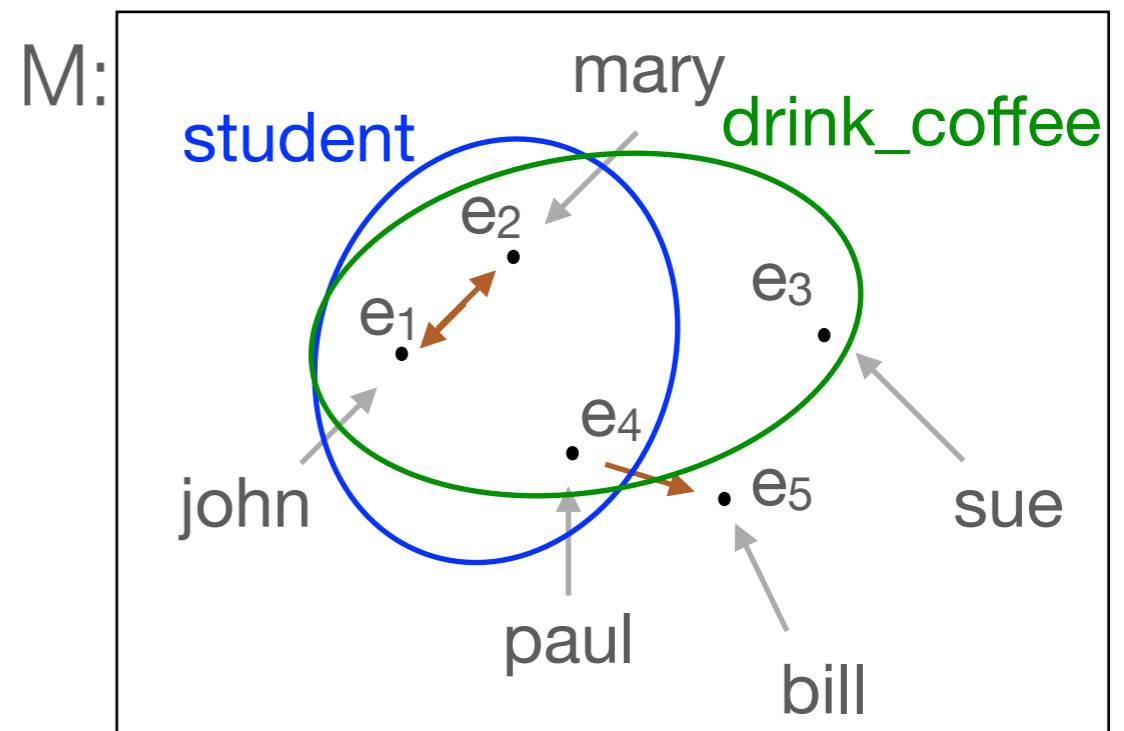
constants

$V_M(\text{bill}) = e_5$

$V_M(\text{student}) = \{e_1, e_2, e_4\}$

$V_M(\text{drink_coffee}) = \{e_1, e_2, e_3, e_4\}$

$V_M(\text{love}) = \{\langle e_1, e_2 \rangle, \langle e_2, e_1 \rangle, \langle e_4, e_5 \rangle\}$



1-place predicates

2-place predicates

Interpretation in the model

V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n -ary relations over U_M to n -place predicate symbols:

- $V_M(c) \in U_M$ if c is an individual constant
- $V_M(P) \subseteq U_M^n$ if P is an n -place predicate symbol
- $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol

Variables and quantifiers

How to interpret the following sentence in our model M:

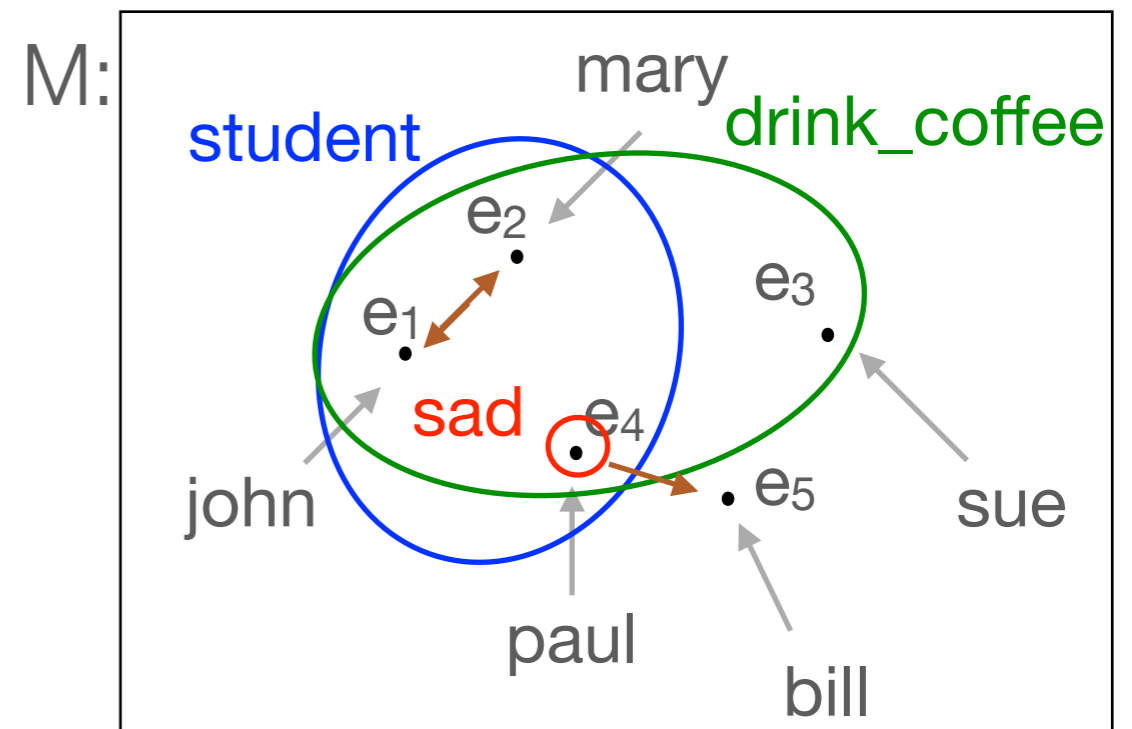
- Someone is sad $\mapsto \exists x(\text{sad}'(x))$

Intuition:

- find an entity in the universe for which the statement holds: $V_M(\text{sad}') = e_4$
- replace x by e_4 in order to make $\exists x(\text{sad}'(x))$ true

More formally:

- Interpret sentence relative to *assignment function* g : i.e., $\llbracket \exists x(\text{sad}'(x)) \rrbracket^{M,g}$, such that $g(x) = e_4$; this can be generalised to any g' as follows: $g'[x/e_4](x) = e_4$



Assignment functions

An assignment function g assigns values to all variables

- $g :: \text{VAR} \rightarrow U_M$
- We write $g[x/d]$ for the assignment function g' that assigns d to x and assigns the same values as g to all other variables.

	x	y	z	u	...
g	e_1	e_2	e_3	e_4	...
$g[y/e_1]$	e_1	e_1	e_3	e_4	...
$g[x/e_1]$	e_1	e_2	e_3	e_4	...
$g[y/g(z)]$	e_1	e_3	e_3	e_4	...
$g[y/e_1][u/e_1]$	e_1	e_1	e_3	e_1	...
$g[y/e_1][y/e_2]$	e_1	e_2	e_3	e_4	...

Interpretation of terms

Interpretation of terms with respect to a model M and a variable assignment g :

$$\llbracket \alpha \rrbracket^{M,g} = \begin{array}{ll} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{array}$$

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g :

- $\llbracket R(t_1, \dots, t_n) \rrbracket^{M,g} = 1$ iff $\langle \llbracket t_1 \rrbracket^{M,g}, \dots, \llbracket t_n \rrbracket^{M,g} \rangle \in V_M(R)$
- $\llbracket t_1 = t_2 \rrbracket^{M,g} = 1$ iff $\llbracket t_1 \rrbracket^{M,g} = \llbracket t_2 \rrbracket^{M,g}$
- $\llbracket \neg\phi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 0$
- $\llbracket \phi \wedge \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 1$ and $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \vee \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 1$ or $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \rightarrow \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = 0$ or $\llbracket \psi \rrbracket^{M,g} = 1$
- $\llbracket \phi \leftrightarrow \psi \rrbracket^{M,g} = 1$ iff $\llbracket \phi \rrbracket^{M,g} = \llbracket \psi \rrbracket^{M,g}$
- $\llbracket \exists x\phi \rrbracket^{M,g} = 1$ iff there is a $d \in U_M$ such that $\llbracket \phi \rrbracket^{M,g[x/d]} = 1$
- $\llbracket \forall x\phi \rrbracket^{M,g} = 1$ iff for all $d \in U_M$, $\llbracket \phi \rrbracket^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula ϕ is true in a model M iff:

$\llbracket \phi \rrbracket^{M,g} = 1$ for every variable assignment g

A formula ϕ is valid ($\models \phi$) iff:

ϕ is true in all models

A formula ϕ is satisfiable iff:

there is at least one model M such that ϕ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:

there is a model M such that every formula in Γ is true in M
("M satisfies Γ ," or "M is a model of Γ ")

Γ entails a formula ϕ ($\Gamma \models \phi$) iff:

ϕ is true in every model structure that satisfies Γ

Logical Equivalence

Formula ϕ is logically equivalent to formula ψ ($\phi \Leftrightarrow \psi$), iff:

- $\llbracket \phi \rrbracket^{M,g} = \llbracket \psi \rrbracket^{M,g}$ for all models M and variable assignments g .

For all *closed* formulas ϕ and ψ , the following assertions are equivalent:

1. $\phi \Leftrightarrow \psi$ (logical equivalence)
2. $\phi \models \psi$ and $\psi \models \phi$ (mutual entailment)
3. $\models \phi \leftrightarrow \psi$ (validity of “material equivalence”)

Logical Equivalence Theorems: Propositions

1) $\neg\neg\phi \Leftrightarrow \phi$

Double negation

2) $\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$

Commutativity of \wedge , \vee

3) $\phi \vee \psi \Leftrightarrow \psi \vee \phi$

4) $\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$

Distributivity of \wedge and \vee

5) $\phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$

6) $\neg(\phi \wedge \psi) \Leftrightarrow \neg\phi \vee \neg\psi$

de Morgan's Laws

7) $\neg(\phi \vee \psi) \Leftrightarrow \neg\phi \wedge \neg\psi$

8) $\phi \rightarrow \neg\psi \Leftrightarrow \psi \rightarrow \neg\phi$

Law of Contraposition

9) $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$

10) $\neg(\phi \rightarrow \psi) \Leftrightarrow \phi \wedge \neg\psi$

Logical Equivalence Theorems: Quantifiers

11) $\neg \forall x \phi \Leftrightarrow \exists x \neg \phi$

Quantifier negation

12) $\neg \exists x \phi \Leftrightarrow \forall x \neg \phi$

13) $\forall x (\phi \wedge \psi) \Leftrightarrow \forall x \phi \wedge \forall x \psi$

Quantifier distribution

14) $\exists x (\phi \vee \psi) \Leftrightarrow \exists x \phi \vee \exists x \psi$

15) $\forall x \forall y \phi \Leftrightarrow \forall y \forall x \phi$

Quantifier Swap

16) $\exists x \exists y \phi \Leftrightarrow \exists y \exists x \phi$

17) $\exists x \forall y \phi \Rightarrow \forall y \exists x \phi$

... but not vice versa !

Logical Equivalence Theorems: Quantifiers (cont.)

The following equivalences are valid theorems of FOL, provided that x does not occur free in ϕ :

Here, $\phi[x/y]$ is the result of replacing all free occurrences of y in ϕ with x

$$18) \exists y\phi \Leftrightarrow \exists x\phi[x/y]$$

$$19) \forall y\phi \Leftrightarrow \forall x\phi[x/y]$$

$$20) \phi \wedge \forall x\Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$$

$$21) \phi \wedge \exists x\Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$$

$$22) \phi \vee \forall x\Psi \Leftrightarrow \forall x(\phi \vee \Psi)$$

$$23) \phi \vee \exists x\Psi \Leftrightarrow \exists x(\phi \vee \Psi)$$

$$24) \phi \rightarrow \forall x\Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$$

$$25) \phi \rightarrow \exists x\Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$$

$$26) \exists x\Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$$

$$27) \forall x\Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$$

cf. ex. 7
on slide 11!

Equivalence Transformations

(1) $\neg \exists x \forall y (Py \rightarrow Rxy)$ “Nobody masters every problem”

(2) $\forall x \exists y (Py \wedge \neg Rxy)$ “Everybody fails to master some problem”

We show the equivalence of (1) and (2) as follows:

$$\begin{aligned} \neg \exists x \forall y (Py \rightarrow Rxy) &\Leftrightarrow \forall x \neg \forall y (Py \rightarrow Rxy) && (\neg \exists x \phi \Leftrightarrow \forall x \neg \phi) \\ &\Leftrightarrow \forall x \exists y \neg (Py \rightarrow Rxy) && (\neg \forall x \phi \Leftrightarrow \exists x \neg \phi) \\ &\Leftrightarrow \forall x \exists y (Py \wedge \neg Rxy) && (\neg(\phi \rightarrow \psi) \Leftrightarrow \phi \wedge \neg \psi) \end{aligned}$$

Background reading material

- Gamut: Logic, Language, and Meaning Vol I — Chapter 3
- Winter: Elements of Formal Semantics — Chapter 2
<http://www.phil.uu.nl/~yoad/efs/main.html>
- For a more basic introduction, see:
<http://www.logicinaction.org> — Chapter 4