Semantic Theory
Week 2 – Predicate Logic

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Information about this course

Contact information:

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Recommended literature:

- Gamut: Logic, Language, and Meaning, Vol. 2, University of Chicago Press, 1991

- Kamp and Reyle: From Discourse to Logic, Kluwer, 1993

Final exam:

- Exam date to be confirmed

Part I:
Sentence semantics

Sentence meaning

Truth-conditional semantics:

to know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true:

$$\text{Sentence meaning} = \text{truth-conditions}$$

Indirect interpretation:

1. Translate sentences into logical formulas:
    Every student works $\mapsto$ $\forall x(\text{student'}(x) \rightarrow \text{work'}(x))$

2. Interpret these formulas in a logical model:
    $[\![\forall x(\text{student'}(x) \rightarrow \text{work'}(x))]\!]^{M,g} = 1$ iff $V_M(\text{student'}) \subseteq V_M(\text{work'})$

Step 1: Translation

Limits of propositional logic: propositions with internal structure
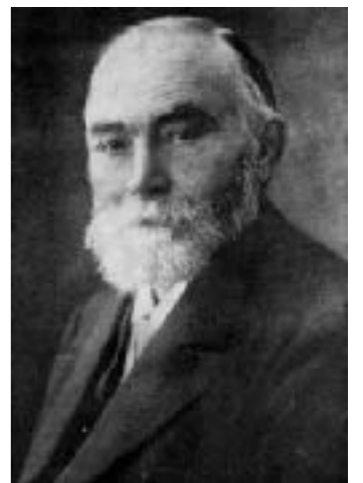
Every man is mortal.

Socrates is a man.

Therefore, Socrates is mortal.

Solution: first-order predicate logic

predicates are expressions
that contain *arguments*
(that can be quantified over)

predication & quantification
over *individuals*

Gottlob Frege

Predicate Logic: Vocabulary

Non-logical expressions:

Individual constants: CON

n-place relation constants: $PRED^n$, for all $n \geq 0$

Infinite set of individual variables: VAR

Logical connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$

Brackets: (,)

Predicate Logic: Syntax

Terms: TERM = VAR ∪ CON

Atomic formulas:

- $R(t_1,…, t_n)$ for $R \in PRED^n$ and $t_1, …, t_n \in TERM$

- $t_1 = t_2$ for $t_1, t_2 \in TERM$

Well-formed formula (WFF):

1. All atomic formulas are WFFs;

2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;

3. If $x \in VAR$, and ϕ is a WFF, then $\forall x \phi$ and $\exists x \phi$ are WFFs;

4. Nothing else is a WFF.

Variable binding

- Given a quantified formula ∀xφ (or ∃xφ), we say that φ (and every part of φ) is in the **scope** of the quantifier ∀x (or ∃x);

- A variable x is **bound** in formula ψ if x occurs in the scope of ∀x or ∃x in ψ;

- If a variable is not bound in formula ψ, it occurs **free** in ψ;

- A **closed formula** is a formula without free variables.

Formalizing Natural Language

1. *Bill loves Mary.*

2. *Bill reads an interesting book.*

3. *Every student reads a book.*

4. *Bill passed every exam.*

5. *Not every student answered every question.*

6. *Only Mary answered every question.*

7. *Mary is annoyed when someone is noisy.*

8. *Although nobody makes noise, Mary is annoyed.*

Step 2: Interpretation

Logical models are simplified representations of the state of affairs in the world
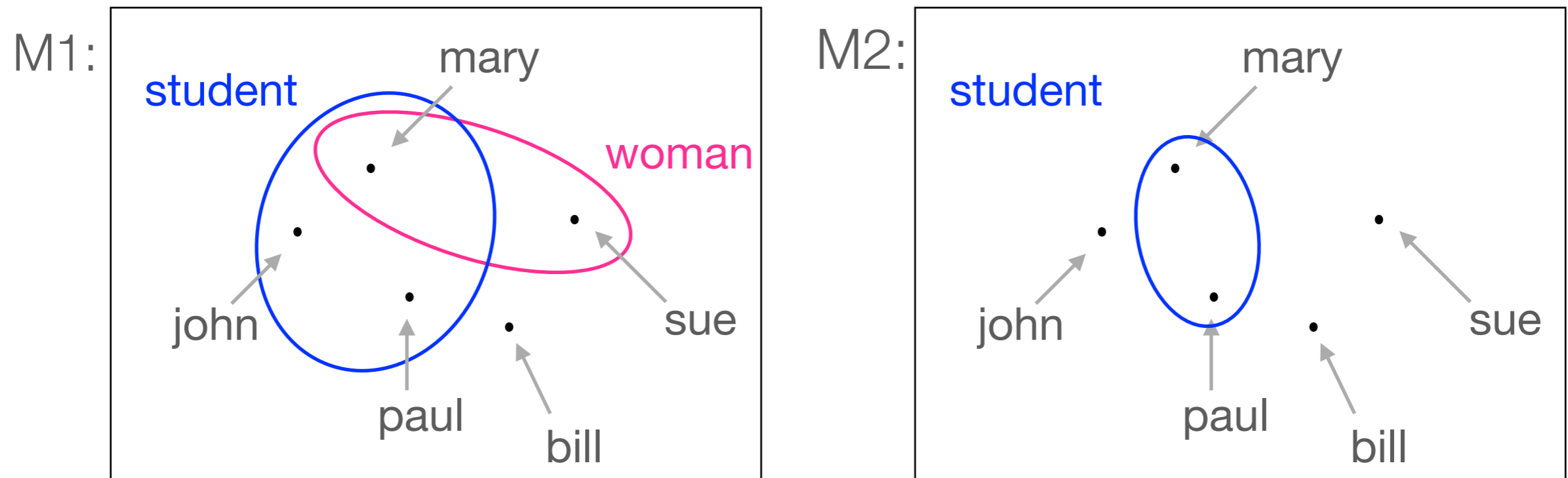
John is a student : for any M, $[\![student'(john)]\!]^{M} = 1$ *iff* $V_M(john) \in V_M(student')$

$V_{M1}(john) \in V_{M1}(student')$ therefore: $[\![student'(john)]\!]^{M1} = 1$
$V_{M2}(john) \notin V_{M2}(student')$ therefore: $[\![student'(john)]\!]^{M2} = 0$

A formal description of a model

Model M = $\langle U_M, V_M \rangle$, with:

- U_M is the universe of M and

- V_M is an interpretation function

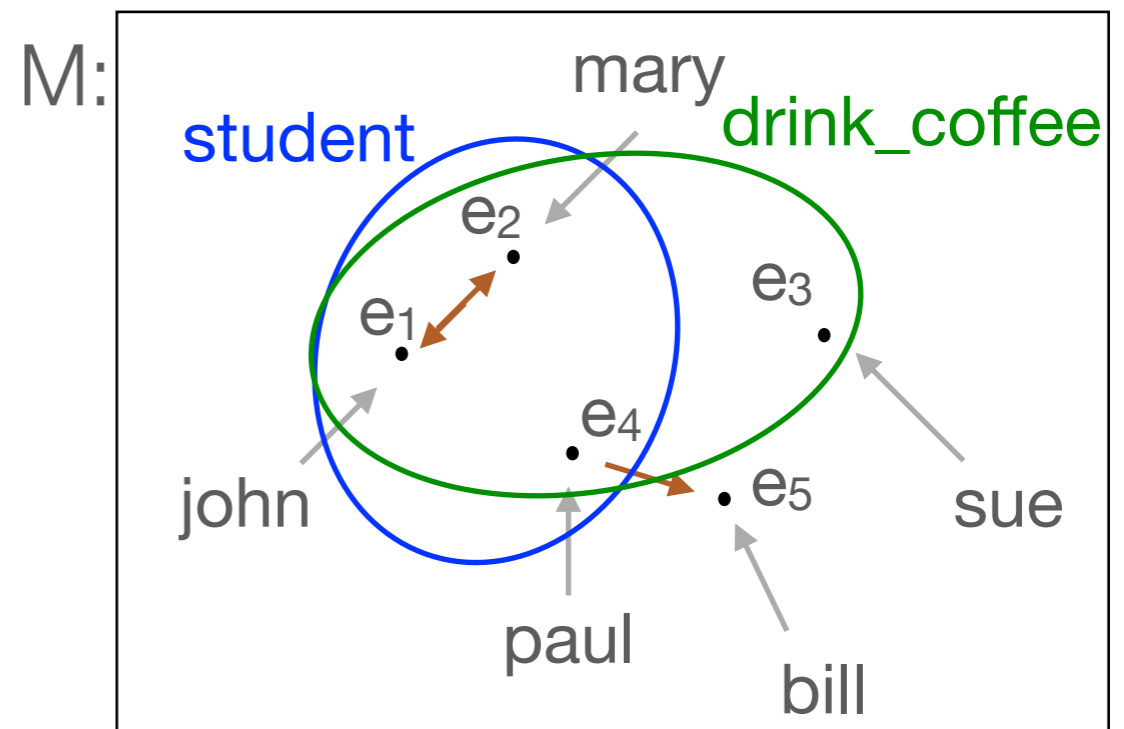U_M = {e1, e2, e3, e4, e5}   **universe**

V_M(john) = e1
…   **constants**
V_M(bill) = e5

V_M(student) = {e1, e2, e4}
V_M(drink_coffee) = {e1, e2, e3, e4}   **1-place predicates**

V_M(love) = {$\langle$e1,e2$\rangle$, $\langle$e2,e1$\rangle$, $\langle$e4,e5$\rangle$}   **2-place predicates**

M: mary
drink_coffee
student
e2
e3
e1
e4
john   e5   sue
paul
bill

Interpretation in the model

V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:

- $V_M(c) \in U_M$ if c is an individual constant

- $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol

- $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol

Variables and quantifiers

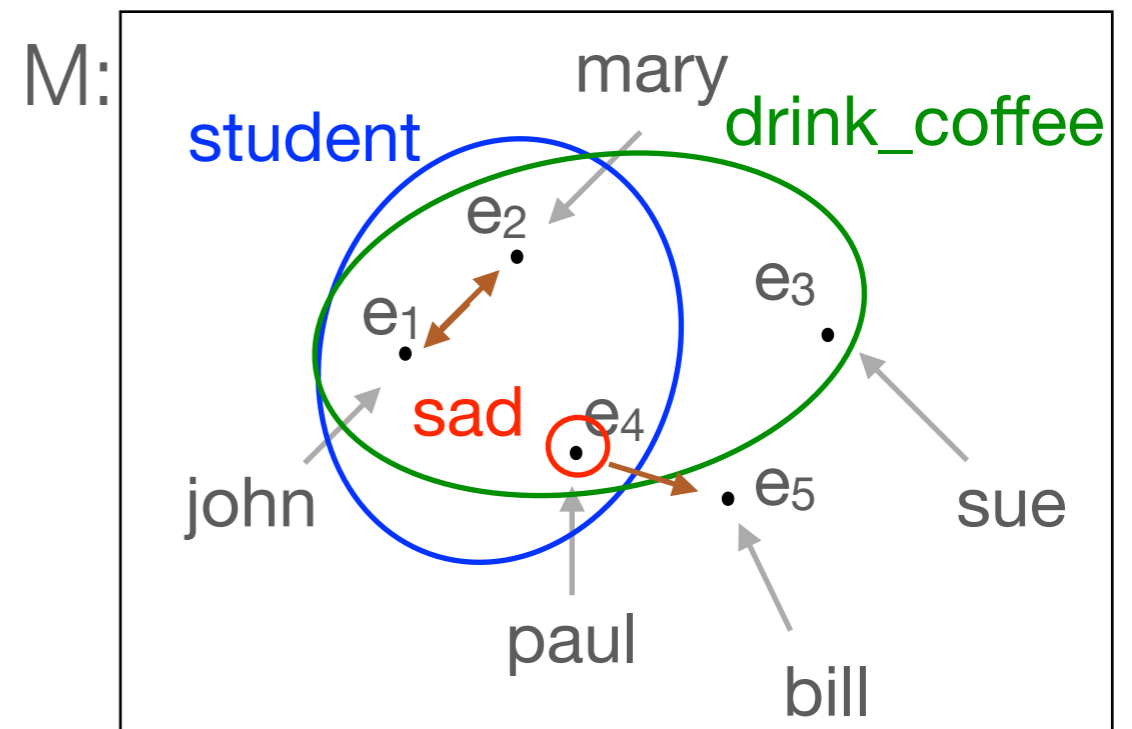How to interpret the following sentence in our model M:

- Someone is sad $\mapsto$ $\exists x(\text{sad'}(x))$

Intuition:

- find an entity in the universe for which the statement holds: $V_M(\text{sad'}) = e_4$

- replace x by e_4 in order to make $\exists x(\text{sad'}(x))$ true

More formally:

- Interpret sentence relative to *assignment function* g: i.e., $[\![\exists x(\text{sad'}(x))]\!]^{M,g}$, such that $g(x) = e_4$; this can be generalised to any g' as follows: $g'[x/e_4](x) = e_4$

M:

Assignment functions

An assignment function **g** assigns values to all variables

- $g :: VAR \rightarrow U_M$

- We write *g[x/d]* for the assignment function *g'* that assigns *d* to *x* and assigns the same values as *g* to all other variables.

	x	y	z	u	…
g	e_1	e_2	e_3	e_4	…
g[y/e_1]	e_1	e_1	e_3	e_4	…
g[x/e_1]	e_1	e_2	e_3	e_4	…
g[y/g(z)]	e_1	e_3	e_3	e_4	…
g[y/e_1][u/e_1]	e_1	e_1	e_3	e_1	…
g[y/e_1][y/e_2]	e_1	e_2	e_3	e_4	…

Interpretation of terms

Interpretation of terms with respect to a model *M* and a variable assignment *g*:

$$[\![α]\!]^{M,g} = \quad V_M(α) \quad \text{if α is an individual constant}$$

$$g(α) \quad \text{if α is a variable}$$

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g:

- $[\![R(t_1, ..., t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$

- $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$

- $[\![\neg \phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$

- $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

- $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

- $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

- $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

- $[\![\exists x \phi]\!]^{M,g} = 1$ iff   there is a $d \in U_M$ such that $[\![\phi]\!]^{M,g[x/d]} = 1$

- $[\![\forall x \phi]\!]^{M,g} = 1$ iff   for all $d \in U_M$, $[\![\phi]\!]^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula φ is true in a model M iff:
$[\![φ]\!]^{M,g} = 1$ for every variable assignment g

A formula φ is valid ($\models$ φ) iff:
φ is true in all models

A formula φ is satisfiable iff:
there is at least one model M such that φ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:
there is a model M such that every formula in Γ is true in M
("M satisfies Γ," or "M is a model of Γ")

Γ entails a formula φ (Γ $\models$ φ) iff:
φ is true in every model structure that satisfies Γ

Logical Equivalence

Formula φ is logically equivalent to formula ψ (φ⇔ψ), iff:

- $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$ for all models M and variable assignments g.

For all *closed* formulas φ and ψ, the following assertions are equivalent:

1. φ⇔ψ   (logical equivalence)

2. φ ⊨ ψ and ψ ⊨ φ   (mutual entailment)

3. ⊨ φ ↔ ψ   (validity of "material equivalence")

Logical Equivalence Theorems: Propositions

1) ¬¬φ ⇔ φ   Double negation

2) φ∧ψ ⇔ ψ∧φ   Commutativity of ∧, ∨

3) φ∨ψ ⇔ ψ∨φ

4) φ∧(ψ∨χ) ⇔ (φ∧ψ)∨(φ∧χ)   Distributivity of ∧ and ∨

5) φ∨(ψ∧χ) ⇔ (φ∨ψ)∧(φ∨χ)

6) ¬(φ∧ψ) ⇔ ¬φ∨¬ψ   de Morgan's Laws

7) ¬(φ∨ψ) ⇔ ¬φ∧¬ψ

8) φ →¬ψ ⇔ ψ →¬φ   Law of Contraposition

9) φ→ψ ⇔ ¬φ∨ψ

10) ¬(φ → ψ) ⇔ φ∧¬ψ

Logical Equivalence Theorems: Quantifiers

11)   ¬∀xφ ⇔ ∃x¬φ   Quantifier negation

12)   ¬∃xφ ⇔ ∀x¬φ

13)   ∀x(φ ∧ Ψ) ⇔ ∀xφ ∧ ∀xΨ   Quantifier distribution

14)   ∃x(φ ∨ Ψ) ⇔ ∃xφ ∨ ∃xΨ

15)   ∀x∀yφ ⇔ ∀y∀xφ   Quantifier Swap

16)   ∃x∃yφ ⇔ ∃y∃xφ

17)   ∃x∀yφ ⇒ ∀y∃xφ   … but not vice versa !

Logical Equivalence Theorems: Quantifiers (cont.)

The following equivalences are valid theorems of FOL, provided that x does not occur free in φ:

Here, φ[x/y] is the result of replacing all free occurrences of y in φ with x

18) $\exists y \phi \Leftrightarrow \exists x \phi[x/y]$

19) $\forall y \phi \Leftrightarrow \exists x \phi[x/y]$

20) $\phi \wedge \forall x \Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$

21) $\phi \wedge \exists x \Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$

22) $\phi \vee \forall x \Psi \Leftrightarrow \forall x(\phi \vee \Psi)$

23) $\phi \vee \exists x \Psi \Leftrightarrow \exists x(\phi \vee \Psi)$

24) $\phi \rightarrow \forall x \Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$

25) $\phi \rightarrow \exists x \Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$

26) $\exists x \Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$

27) $\forall x \Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$

Equivalence Transformations

(1) ¬∃x∀y(Py → Rxy)   "Nobody masters every problem"

(2) ∀x∃y(Py ∧ ¬Rxy)   "Everybody fails to master some problem"

We show the equivalence of (1) and (2) as follows:

¬∃x∀y(Py → Rxy)   ⇔ ∀x¬∀y(Py → Rxy)   (¬∃xφ ⇔ ∀x¬φ)

⇔ ∀x∃y¬(Py → Rxy)   (¬∀xφ ⇔ ∃x¬φ)

⇔ ∀x∃y(Py ∧ ¬Rxy)   (¬(φ → ψ) ⇔ φ∧¬ψ)

Background reading material

- Gamut: Logic, Language, and Meaning Vol I/II — Chapter 2

- For a more basic introduction, see:
  http://www.logicinaction.org — Chapter 4

5. Not every student answered every question.

- ¬∀x(S(x) → ∀y(Q(y) → A(x,y)))
- ¬∀x∀y(S(x) → (Q(y) → A(x,y)))   *(24)*
- ¬∀x∀y((S(x) ∧ Q(y)) → A(x,y))   *(*)*
- ∃x¬∀y((S(x) ∧ Q(y)) → A(x,y))   *(11)*
- ∃x∃y¬((S(x) ∧ Q(y)) → A(x,y))   *(11)*
- ∃x∃y((S(x) ∧ Q(y)) ∧ ¬A(x,y))   *(10)*
- ∃x∃y((Q(y) ∧ S(x)) ∧ ¬A(x,y))   *(2)*
- ∃x∃y(Q(y) ∧ (S(x) ∧ ¬A(x,y)))   *(associativity)*
- ∃y∃x(Q(y) ∧ (S(x) ∧ ¬A(x,y)))   *(16)*
- ∃y(Q(y) ∧ ∃x(S(x) ∧ ¬A(x,y)))   *(21)*
- ∃y(Q(y) ∧ ∃x¬(S(x) → A(x,y)))   *(10)*
- ∃y(Q(y) ∧ ¬∀x(S(x) → A(x,y)))   *(11)*
- ∃x(Q(x) ∧ ¬∀y(S(y) → A(y,x)))   *(18+19)*

p → (q → r)		
p → (¬q ∨ r)	(9)	
¬p ∨ (¬q ∨ r)	(9)	
(¬p ∨ ¬q) ∨ r	(3)	
¬(p ∧ q) ∨ r	(6)	
(p ∧ q) → r	(9)	

p	q	r	(p ∧ q) ∧ r	p ∧ (q ∧ r)
1	1	1	1	1
1	1	0	0	0
1	0	1	0	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0