Semantic Theory
Week 4 – Typed Lambda Calculus

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Type Theory — Syntax

For every type τ, the set of well-formed expressions WE_τ is defined as follows:

(i) $CON_\tau \subseteq WE_\tau$ and $VAR_\tau \subseteq WE_\tau$;

(ii) If $\alpha \in WE_{\langle\sigma, \tau\rangle}$, and $\beta \in WE_\sigma$, then $\alpha(\beta) \in WE_\tau$;
(function application)

(iii) If A, B are in WE_t, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are in WE_t;

(iv) If A is in WE_t and x is a variable of arbitrary type, then $\forall xA$ and $\exists xA$ are in WE_t;

(v) If α, β are well-formed expressions of the same type, then $\alpha = \beta \in WE_t$;

(vi) Nothing else is a well-formed formula.

Function application

(ii) If $\alpha \in WE_{\langle\sigma,\tau\rangle}$, and $\beta \in WE_{\sigma}$, then $\alpha(\beta) \in WE_{\tau}$

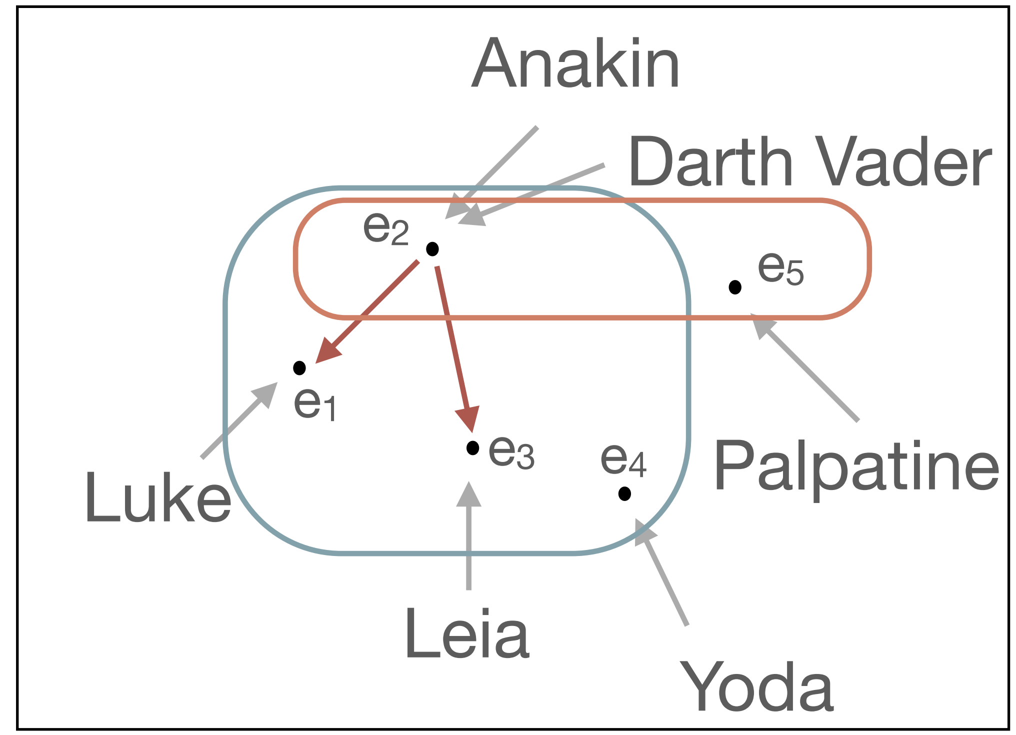
“John is a talented piano player”

$$\dfrac{\text{john} :: e \qquad \dfrac{\text{piano\_player} :: \langle e, t\rangle \qquad \text{talented} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented(piano\_player)} :: \langle e, t\rangle}}{\text{talented(piano\_player)(john)} :: t}$$

Type Theory — Model

Consider the following Model M:

M:



$D_e = U_M = \{e_1, e_2, e_3, e_4, e_5\}$

$V_M(\text{Anakin}_e) = V_M(\text{Darth Vader}_e) = e_2$

$V_M(\text{Yedi}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \rightarrow 1 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 1 \\ e_4 \rightarrow 1 \\ e_5 \rightarrow 0 \end{bmatrix}$ $V_M(\text{Dark\_Sider}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \rightarrow 0 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 0 \\ e_4 \rightarrow 0 \\ e_5 \rightarrow 1 \end{bmatrix}$

$$V_M(\text{Powerful}_{\langle\langle e,t\rangle\langle e,t\rangle\rangle}) = \begin{bmatrix} \begin{bmatrix} e_1 \rightarrow 1 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 1 \\ e_4 \rightarrow 1 \\ e_5 \rightarrow 0 \end{bmatrix} \rightarrow \begin{bmatrix} e_1 \rightarrow 0 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 0 \\ e_4 \rightarrow 1 \\ e_5 \rightarrow 0 \end{bmatrix} \\ \begin{bmatrix} e_1 \rightarrow 0 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 0 \\ e_4 \rightarrow 0 \\ e_5 \rightarrow 1 \end{bmatrix} \rightarrow \begin{bmatrix} e_1 \rightarrow 0 \\ e_2 \rightarrow 1 \\ e_3 \rightarrow 0 \\ e_4 \rightarrow 0 \\ e_5 \rightarrow 1 \end{bmatrix} \\ \dots \end{bmatrix}$$

$\dashrightarrow$ Powerful $X_{\langle e,t\rangle} \vDash X_{\langle e,t\rangle}$

Type Theory — Interpretation

Interpretation with respect to a model structure $M = \langle U, V \rangle$ and a variable assignment g:

- $[\![\alpha]\!]^{M,g} = V(\alpha)$ if α is a constant
  $[\![\alpha]\!]^{M,g} = g(\alpha)$ if α is a variable
- $[\![\alpha(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
  $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
  $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
  …
- $[\![\alpha = \beta]\!]^{M,g} = 1$ iff $[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}$
- *For any variable v of type σ:*
  $[\![\exists v\phi]\!]^{M,g} = 1$ iff there is a $d \in D_\sigma$ such that $[\![\phi]\!]^{M,g[v/d]} = 1$
  $[\![\forall v\phi]\!]^{M,g} = 1$ iff for all $d \in D_\sigma : [\![\phi]\!]^{M,g[v/d]} = 1$

Compositionality

The principle of compositionality: “The meaning of a complex expression is a function of the meanings of its parts and of the syntactic rules by which they are combined” (Partee et al.,1993)

Compositional semantics construction:

- compute meaning representations for sub-expressions
- combine them to obtain a meaning representation for a complex expression.

Problematic case: “Not smoking$_{\langle e,t\rangle}$ is healthy$_{\langle\langle e,t\rangle,t\rangle}$”

!

Lambda abstraction

λ-abstraction is an operation that takes an expression and "opens" specific argument positions.

Syntactic definition:

If α is in WE_τ, and x is in VAR_σ then $\lambda x(\alpha)$ is in $WE_{\langle\sigma, \tau\rangle}$

- The scope of the λ-operator is the smallest WE to its right. Wider scope must be indicated by brackets.
- We often use the "dot notation" $\lambda x.\phi$ indicating that the λ-operator takes widest possible scope (over ϕ).

Interpretation of Lambda-expressions

If $\alpha \in WE_\tau$ and $v \in VAR_\sigma$, then $[\![\lambda v \alpha]\!]^{M,g}$ is that function $f : D_\sigma \rightarrow D_\tau$ such that for all $a \in D_\sigma$, $f(a) = [\![\alpha]\!]^{M,g[v/a]}$

If the λ-expression is applied to some argument, we can simplify the interpretation:

- $[\![\lambda v \alpha]\!]^{M,g}(x) = [\![\alpha]\!]^{M,g[v/x]}$

Example: *"Bill is a non-smoker"*

$[\![\lambda x(\neg S(x))(b')]\!]^{M,g} = 1$

iff $[\![\lambda x(\neg S(x))]\!]^{M,g}([\![b']\!]^{M,g}) = 1$

iff $[\![\neg S(x)]\!]^{M,g[x/[\![b']\!]^{M,g}]} = 1$

iff $[\![S(x)]\!]^{M,g[x/[\![b']\!]^{M,g}]} = 0$

iff $[\![S]\!]^{M,g[x/[\![b']\!]^{M,g}]}([\![x]\!]^{M,g[x/[\![b']\!]^{M,g}]}) = 0$

iff $V_M(S)(V_M(b')) = 0$

β-Reduction

$[\![\lambda v(\alpha)(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g[v/[\![\beta]\!]^{M,g}]}$

⇒ all (free) occurrences of the λ-variable in α get the interpretation of β as value.

This operation is called β-reduction

- $\lambda v(\alpha)(\beta) \Leftrightarrow [\beta/v]\alpha$
- $[\beta/v]\alpha$ is the result of replacing all free occurrences of v in α with β.

Achtung: The equivalence is not unconditionally valid!

Variable capturing

Q: Are $\lambda v(\alpha)(\beta)$ and $[\beta/v]\alpha$ always equivalent?

- $\lambda x(\text{drive'}(x) \wedge \text{drink'}(x))(j') \Leftrightarrow \text{drive'}(j') \wedge \text{drink'}(j')$
- $\lambda x(\text{drive'}(x) \wedge \text{drink'}(x))(y) \Leftrightarrow \text{drive'}(y) \wedge \text{drink'}(y)$
- $\lambda x(\forall y\ \text{know'}(x)(y))(j') \Leftrightarrow \forall y\ \text{know}(j')(y)$
- NOT: $\lambda x(\forall y\ \text{know'}(x)(y))(y) \Leftrightarrow \forall y\ \text{know}(y)(y)$

Let v, v' be variables of the same type, and let α be any well-formed expression.

- v is free for v' in α iff no free occurrence of v' in α is in the scope of a quantifier or a λ-operator that binds v.

Conversion rules

- β-conversion: $\lambda v(\alpha)(\beta) \Leftrightarrow [\beta/v]\alpha$

  (if all free variables in β are free for v in α)

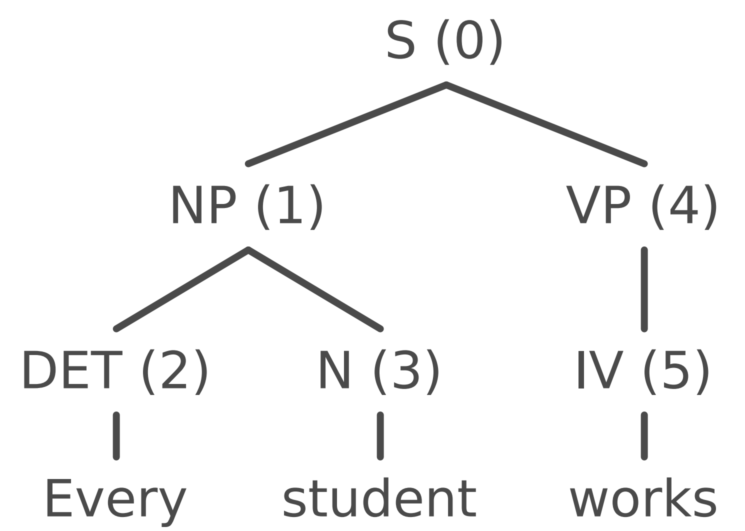
- α-conversion: $\lambda v\alpha \Leftrightarrow \lambda w[w/v]\alpha$

  (if w is free for v in α)

- η-conversion: $\lambda v(\alpha(v)) \Leftrightarrow \alpha$

β-Reduction Example

Every student works.



(2) $\lambda P \lambda Q \forall x(P(x) \rightarrow Q(x)) : \langle\langle e, t\rangle, \langle\langle e, t\rangle, t\rangle$

(3) $\text{student}' : \langle e, t\rangle$

(1) $\lambda P \lambda Q \forall x(P(x) \rightarrow Q(x))(\text{student}')$

$\Rightarrow^{\beta} \lambda Q \forall x(\text{student}'(x) \rightarrow Q(x)) : \langle\langle e, t\rangle, t\rangle$

(4)/(5) $\text{work}' : \langle e, t\rangle$

(0) $\lambda Q \forall x(\text{student}'(x) \rightarrow Q(x))(\text{work}')$

$\Rightarrow^{\beta} \forall x(\text{student}'(x) \rightarrow \text{work}'(x)) : t$

Background reading material

- Gamut: Logic, Language, and Meaning Vol II
  — Chapter 4 (minus 4.3)