Semantic Theory
Week 3 – Type Theory

Noortje Venhuizen

Universität des Saarlandes

Summer 2016

First-order logic

First-order logic talks about:

- Individual objects
- Properties of and relations between individual objects
- Generalization across individual objects (quantification)

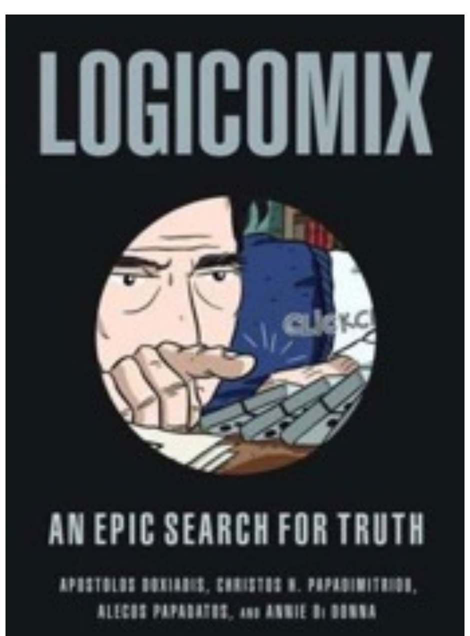
Limitations of first-order logic

FOL is not expressive enough to capture all meanings that can be expressed by basic natural language expressions:

Jumbo is a small elephant.	(Predicate modifiers)
Blond is a hair color.	(Second-order predicates)
Yesterday, it rained.	(Non-logical sentence operators)
Bill and John have the same hair color.	(Higher-order quantification)

What logical system can we use to capture this diversity?

Bertrand Russell



Q: Does the barber shave himself?

Russell's paradox

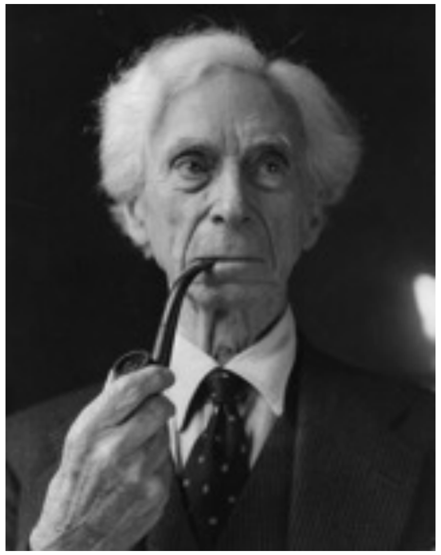
What if we extend the FOL interpretation of predicates, and interpret higher-order predicates as sets of sets of properties?

For every predicate P, we can define a set $\{x \mid P(x)\}$ containing all and only those entities for which P holds.

Then we can define a set $S = \{X \mid X \notin X\}$ representing the set of all sets that are not members of itself.

Q: does S belong to itself?

… we need a more restricted way of talking about *properties* and *relations between properties*!



Type Theory

Basic types:

- **e** – the type of individual terms (“entities”)
- **t** – the type of formulas (“truth-values”)

Complex types:

- If $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ are types, then $\langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle$ is a type
  (representing a functor expression that takes a $\boldsymbol{\sigma}$ type expression as argument and returns a type $\boldsymbol{\tau}$ expression; sometimes written as: $(\boldsymbol{\sigma} \rightarrow \boldsymbol{\tau})$).

Types & Function Application

Types of first-order expressions:

- Individual constants (Luke, Saarbrücken) : e
- One-place predicates (sleep, walk): $\langle e, t \rangle$
- Two-place predicates (read, admire): $\langle e, \langle e, t \rangle\rangle$
- Three-place predicates (give, introduce): $\langle e, \langle e, \langle e, t \rangle\rangle\rangle$

Function application: Combining a functor of complex type with an appropriate argument, resulting in an expression of a less complex type: $\langle \alpha, \beta \rangle(\alpha) \mapsto \beta$

- sleep'(john') :: $\langle e, t \rangle(e) \Longrightarrow t$
- admire'(john') :: $\langle e, \langle e, t \rangle\rangle(e) \Longrightarrow \langle e, t \rangle$

More examples of types

Types of higher-order expressions:

- Predicate modifiers (expensive, poor): $\langle\langle e, t\rangle, \langle e, t\rangle\rangle$
- Second-order predicates (hair colour): $\langle\langle e, t\rangle, t\rangle$
- Sentence operators (yesterday, possibly, unfortunately): $\langle t, t\rangle$
- Degree particles (very, too): $\langle\langle\langle e, t\rangle, \langle e, t\rangle\rangle, \langle\langle e, t\rangle, \langle e, t\rangle\rangle\rangle$

Tip: If σ, τ are basic types, $\langle\sigma, \tau\rangle$ can be abbreviated as $\sigma\tau$. Thus, the type of predicate modifiers and second-order predicates can be more conveniently written as $\langle et, et\rangle$ and $\langle et, t\rangle$, respectively.

Type Theory — Vocabulary

Non-logical constants:

- For every type τ a (possibly empty) set of non-logical constants CON_τ (pairwise disjoint)

Variables:

- For every type τ an infinite set of variables VAR_τ (pairwise disjoint)

Logical symbols: $\forall$, $\exists$, $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $=$

Brackets: (,)

Type Theory — Syntax

For every type τ, the set of well-formed expressions WE_τ is defined as follows:

(i) $CON_\tau \subseteq WE_\tau$ and $VAR_\tau \subseteq WE_\tau$;

(ii) If $\alpha \in WE_{\langle\sigma, \tau\rangle}$, and $\beta \in WE_\sigma$, then $\alpha(\beta) \in WE_\tau$;
(function application)

(iii) If A, B are in WE_t, then $\neg A$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are in WE_t;

(iv) If A is in WE_t and x is a variable of arbitrary type, then $\forall xA$ and $\exists xA$ are in WE_t;

(v) If α, β are well-formed expressions of the same type, then $\alpha = \beta \in WE_t$;

(vi) Nothing else is a well-formed formula.

Function application

(ii) If $\alpha \in WE_{\langle\sigma,\tau\rangle}$, and $\beta \in WE_{\sigma}$, then $\alpha(\beta) \in WE_{\tau}$

"John is a talented piano player"

$$\dfrac{\text{john} :: e \qquad \dfrac{\text{piano\_player} :: \langle e, t\rangle \qquad \text{talented} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented(piano\_player)} :: \langle e, t\rangle}}{\text{talented(piano\_player)(john)} :: t}$$

Higher-order predicates

Higher-order quantification:

- *Bill has the same hair colour as John*

$$\exists C\ (\text{hair\_colour}(C) \wedge C(\text{bill}) \wedge C(\text{john}))$$

$\langle\langle e, t\rangle, t\rangle$ $\langle e, t\rangle$ e

Higher-order equality:

- For p, q ∈ CON_t, "p=q" expresses material equivalence: "p ↔ q".
- For F, G ∈ $CON_{\langle e,\, t\rangle}$, "F=G" expresses co-extensionality: "∀x(Fx↔Gx)"
- For any formula ϕ of type *t*, ϕ=(x=x) is a representation of "ϕ is true".

Type Theory — Semantics [1]

Let **U** be a non-empty set of entities.

The domain of possible denotations $\mathbf{D_\tau}$ for every type $\boldsymbol{\tau}$ is given by:

- $D_e = U$
- $D_t = \{0,1\}$
- $D_{\langle\sigma,\tau\rangle}$ is the set of all functions from D_σ to D_τ

Expressions of type $\boldsymbol{\sigma}$ *denote elements of* $\mathbf{D_\sigma}$

Characteristic functions

Many natural language expressions have a type ⟨**σ, t**⟩

Expressions with type ⟨**σ, t**⟩ are functions mapping elements of type **σ** to truth values: {0,1}

Such functions with a range of {0,1} are called characteristic functions, because they uniquely specify a subset of their domain $\mathbf{D_{\sigma}}$

The characteristic function of set M in a domain U is the function F_M: U→{0,1} such that for all a ∈ U, F_M(a) = 1 iff a ∈ M.

NB: For first-order predicates, the FOL representation (using sets) and the type-theoretic representation (using characteristic functions) are equivalent.

Interpretation with characteristic functions: example

For M = ⟨U, V⟩, let U consist of the persons John, Bill, Mary, Paul, and Sally. For selected types, we have the following sets of possible denotations:

- $D_t = \{0,1\}$
- $D_e = U = \{j, b, m, p, s\}$
- $$D_{<e,t>} = \left\{ \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 1 \\ m \rightarrow 0 \\ p \rightarrow 1 \\ s \rightarrow 1 \end{bmatrix}, \begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 1 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 0 \end{bmatrix}, \ldots \right\}$$

Alternative set notation: $D_{<e,t>} = \{\{j,m,s\},\{j,b,p,s\},\{b,m\},\ldots\}$

Type Theory — Semantics [2]

A model structure for a type theoretic language is a tuple $\mathbf{M} = \langle \mathbf{U}, \mathbf{V} \rangle$ such that:

- **U** is a non-empty domain of individuals
- **V** is an interpretation function, which assigns to every $\boldsymbol{\alpha} \in \mathbf{CON}_{\tau}$ an element of $\mathbf{D}_{\tau}$ (where $\boldsymbol{\tau}$ is an arbitrary type)

The variable assignment function g assigns to every typed variable $\mathbf{v} \in \mathbf{VAR}_{\tau}$ an element of $\mathbf{D}_{\tau}$

Type Theory — Interpretation

Interpretation with respect to a model structure $M = \langle U, V \rangle$ and a variable assignment g:

- $[\![\alpha]\!]^{M,g} = V(\alpha)$ if α is a constant
  $[\![\alpha]\!]^{M,g} = g(\alpha)$ if α is a variable
- $[\![\alpha(\beta)]\!]^{M,g} = [\![\alpha]\!]^{M,g}([\![\beta]\!]^{M,g})$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
  $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
  $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
  …
- $[\![\alpha = \beta]\!]^{M,g} = 1$ iff $[\![\alpha]\!]^{M,g} = [\![\beta]\!]^{M,g}$
- *For any variable v of type σ:*
  $[\![\exists v\phi]\!]^{M,g} = 1$ iff there is a $d \in D_\sigma$ such that $[\![\phi]\!]^{M,g[v/d]} = 1$
  $[\![\forall v\phi]\!]^{M,g} = 1$ iff for all $d \in D_\sigma$: $[\![\phi]\!]^{M,g[v/d]} = 1$

Interpretation: Example

John is a talented piano player

$$\frac{\dfrac{}{\text{john} :: e} \quad \dfrac{\text{piano\_player} :: \langle e, t\rangle \qquad \text{talented} :: \langle\langle e, t\rangle, \langle e, t\rangle\rangle}{\text{talented}(\text{piano\_player}) :: \langle e, t\rangle}}{\text{talented}(\text{piano\_player})(\text{john}) :: t}$$

$[\![\text{talented}(\text{piano\_player})(\text{john})]\!]^{M,g}$

$= [\![\text{talented}(\text{piano\_player})]\!]^{M,g} ([\![\text{john}]\!]^{M,g})$

$= [\![\text{talented}]\!]^{M,g}([\![\text{piano\_player}]\!]^{M,g}) ([\![\text{john}]\!]^{M,g})$

$= V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$

Interpretation: Example (cont.)

$[\![$*John is a talented piano player*$]\!]^{M,g} = V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$

$V_M(\text{john}) = j \ (\in \mathbf{D_e})$

$$V_M(\text{piano\_player}) = \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix} (\in \mathbf{D}_{\langle \mathbf{e,t} \rangle}) \iff \{j,m,s\}$$

$$V_M(\text{talented}) = \begin{bmatrix} \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix} \rightarrow \begin{bmatrix} j \rightarrow 1 \\ b \rightarrow 0 \\ m \rightarrow 0 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix} \\ \begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 0 \\ m \rightarrow 1 \\ p \rightarrow 1 \\ s \rightarrow 1 \end{bmatrix} \rightarrow \begin{bmatrix} j \rightarrow 0 \\ b \rightarrow 0 \\ m \rightarrow 0 \\ p \rightarrow 0 \\ s \rightarrow 1 \end{bmatrix} \\ \ldots \end{bmatrix} (\in \mathbf{D}_{\langle\langle \mathbf{e,t} \rangle \langle \mathbf{e,t} \rangle\rangle}) \iff \begin{bmatrix} \{j,m,s\} \rightarrow \{j,s\} \\ \{m,p,s\} \rightarrow \{s\} \\ \ldots \end{bmatrix}$$

$V_M(\text{talented})(V_M(\text{piano\_player}))$

$V_M(\text{talented})(V_M(\text{piano\_player}))(V_M(\text{john}))$

Defining the right model

Consider the following Model M:

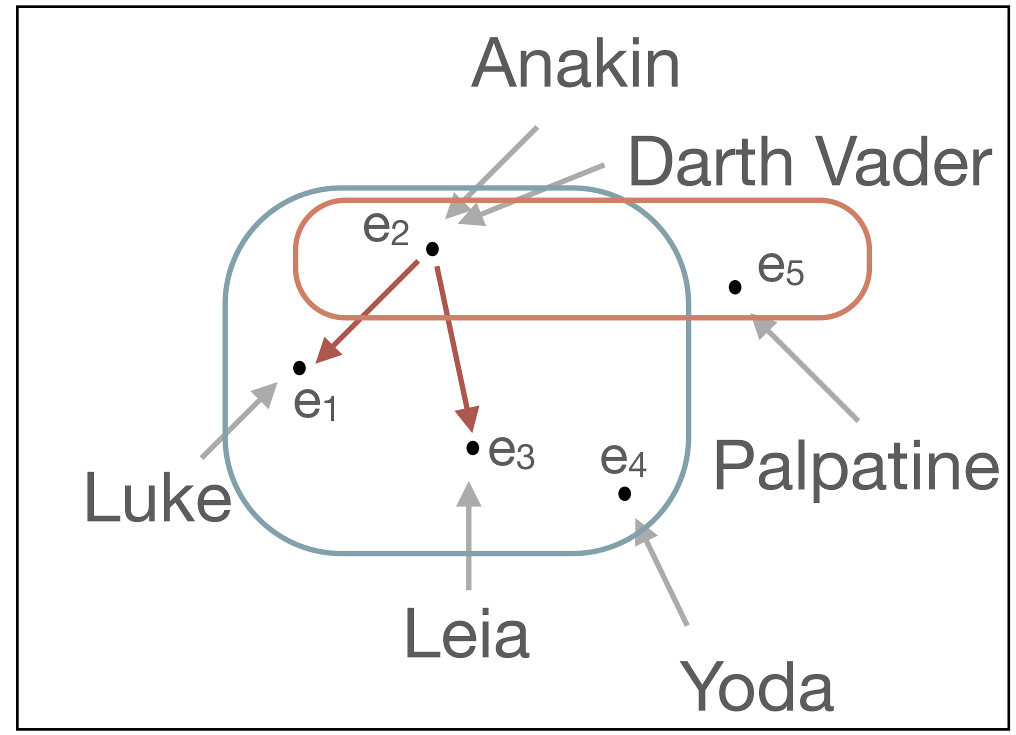
$D_e = U_M = \{e_1, e_2, e_3, e_4, e_5\}$

$V_M(\text{Anakin}_e) = V_M(\text{Darth Vader}_e) = e_2$

$$V_M(\text{Yedi}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix} \quad V_M(\text{Dark\_Sider}_{\langle e,t\rangle}) = \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix}$$

$$V_M(\text{Powerful}_{\langle\langle e,t\rangle\langle e,t\rangle\rangle}) = \begin{bmatrix} \begin{bmatrix} e_1 \to 1 \\ e_2 \to 1 \\ e_3 \to 1 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix} \to \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 1 \\ e_5 \to 0 \end{bmatrix} \\ \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \to \begin{bmatrix} e_1 \to 0 \\ e_2 \to 1 \\ e_3 \to 0 \\ e_4 \to 0 \\ e_5 \to 1 \end{bmatrix} \\ \dots \end{bmatrix}$$

M:



⇢ $\text{Powerful } X_{\langle e,t\rangle} \models X_{\langle e,t\rangle}$

Adjective classes & Meaning postulates

Some valid inferences in natural language:

- Bill is a poor piano player $\models$ Bill is a piano player
- Bill is a blond piano player $\models$ Bill is blond
- Bill is a former professor $\models$ Bill isn't a professor

These entailments do not hold in type theory. Why?

Meaning postulates: restrictions on models which constrain the possible meaning of certain words

Adjective classes & Meaning postulates (cont.)

Restrictive or Subsective adjectives (“poor”)

- $[\![\text{poor N}]\!] \subseteq [\![\text{N}]\!]$
- Meaning postulate: $\forall G \forall x(\text{poor}(G)(x) \rightarrow G(x))$

Intersective adjectives (“blond”)

- $[\![\text{blond N}]\!] = [\![\text{blond}]\!] \cap [\![\text{N}]\!]$
- Meaning postlate: $\forall G \forall x(\text{blond}(G)(x) \rightarrow (\text{blond*}(x) \wedge G(x))$
- NB: $\text{blond} \in WE_{\langle\langle e, t\rangle, \langle e, t\rangle\rangle} \neq \text{blond*} \in WE_{\langle e, t\rangle}$

Privative adjectives (“former”)

- $[\![\text{former N}]\!] \cap [\![\text{N}]\!] = \varnothing$
- Meaning postlate: $\forall G \forall x(\text{former}(G)(x) \rightarrow \neg G(x))$

Background reading material

- Gamut: Logic, Language, and Meaning Vol II
  — Chapter 4 (minus 4.3)