Semantic Theory
Lecture 2 – Predicate Logic

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Information about this course

Contact information:

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Recommended literature:

- Gamut: Logic, Language, and Meaning, Vol. 2, University of Chicago Press, 1991
- Kamp and Reyle: From Discourse to Logic, Kluwer, 1993

Final exam:

- Exam date to be confirmed

Part I:
Sentence semantics

Sentence meaning

Truth-conditional semantics:

to know the meaning of a (declarative) sentence is to know what the world would have to be like for the sentence to be true:

Sentence meaning = truth-conditions

Indirect interpretation:

1. Translate sentences into logical formulas:
   Every student works $\mapsto$ $\forall x(\text{student}'(x) \rightarrow \text{work}'(x))$

2. Interpret these formulas in a logical model:
   $[\![\forall x(\text{student}'(x) \rightarrow \text{work}'(x))]\!]^{M,g} = 1$ iff $V_M(\text{student}') \subseteq V_M(\text{work}')$

Step 1: Translation

Limits of propositional logic: propositions with internal structure

Every man is mortal.

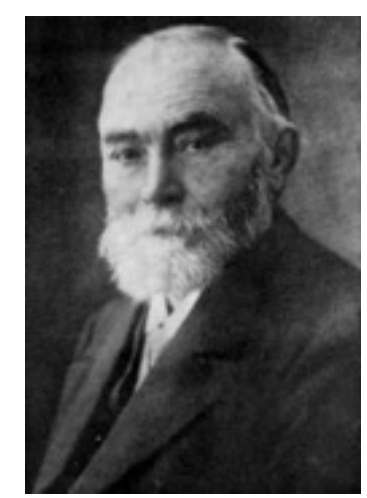
Socrates is a man.

Therefore, Socrates is mortal.

Solution: first-order predicate logic

predicates are expressions
that contain *arguments*
(that can be quantified over)

predication & quantification
over *individuals*



Gottlob Frege

Predicate Logic: Vocabulary

Non-logical expressions:

Individual constants: CON

n-place relation constants: $PRED^n$, for all $n \geq 0$

Infinite set of individual variables: VAR

Logical connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$

Brackets: (,)

Predicate Logic: Syntax

Terms: TERM = VAR $\cup$ CON

Atomic formulas:

- $R(t_1, \ldots, t_n)$ for $R \in \mathrm{PRED}^n$ and $t_1, \ldots, t_n \in$ TERM
- $t_1 = t_2$ for $t_1, t_2 \in$ TERM

Well-formed formula (WFF):

1. All atomic formulas are WFFs;
2. If ϕ and ψ are WFFs, then $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, $(\phi \leftrightarrow \psi)$ are WFFs;
3. If $x \in$ VAR, and ϕ is a WFF, then $\forall x\phi$ and $\exists x\phi$ are WFFs;
4. Nothing else is a WFF.

Variable binding

- Given a quantified formula $\forall x \phi$ (or $\exists x \phi$), we say that ϕ (and every part of ϕ) is in the **scope** of the quantifier $\forall x$ (or $\exists x$);

- A variable x is **bound** in formula ψ if x occurs in the scope of $\forall x$ or $\exists x$ in ψ;

- If a variable is not bound in formula ψ, it occurs **free** in ψ;

- A **closed formula** is a formula without free variables.

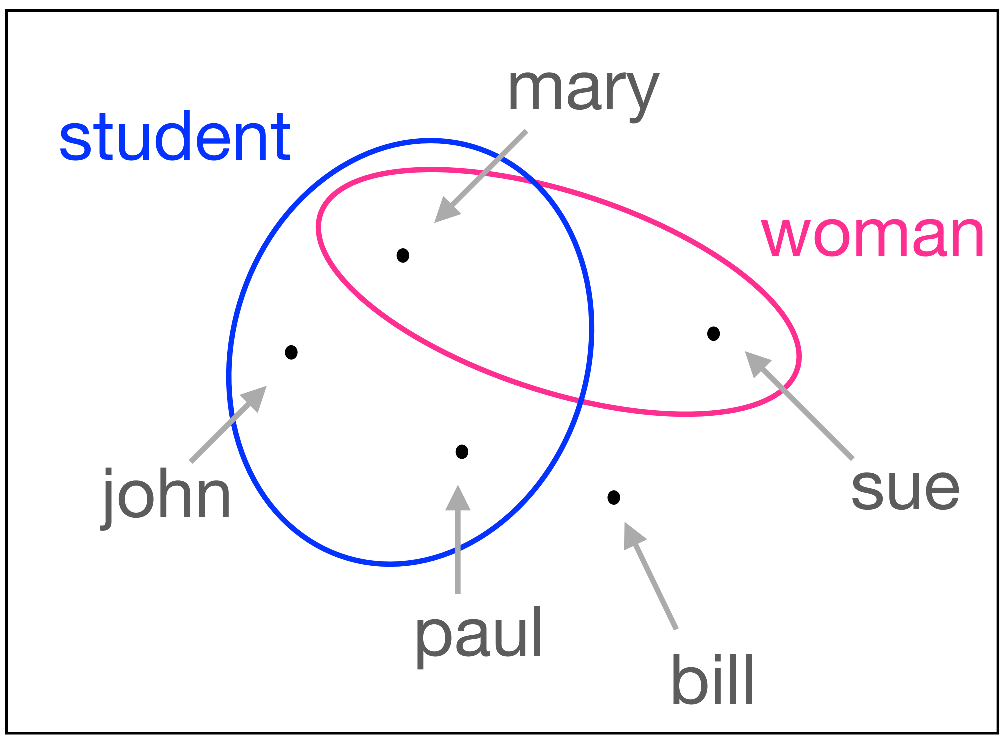
Formalizing Natural Language

1. *Bill loves Mary.*
2. *Bill reads an interesting book.*
3. *Every student reads a book.*
4. *Bill passed every exam.*
5. *Not every student answered every question.*
6. *Only Mary answered every question.*
7. *Mary is annoyed when someone is noisy.*
8. *Although nobody makes noise, Mary is annoyed.*

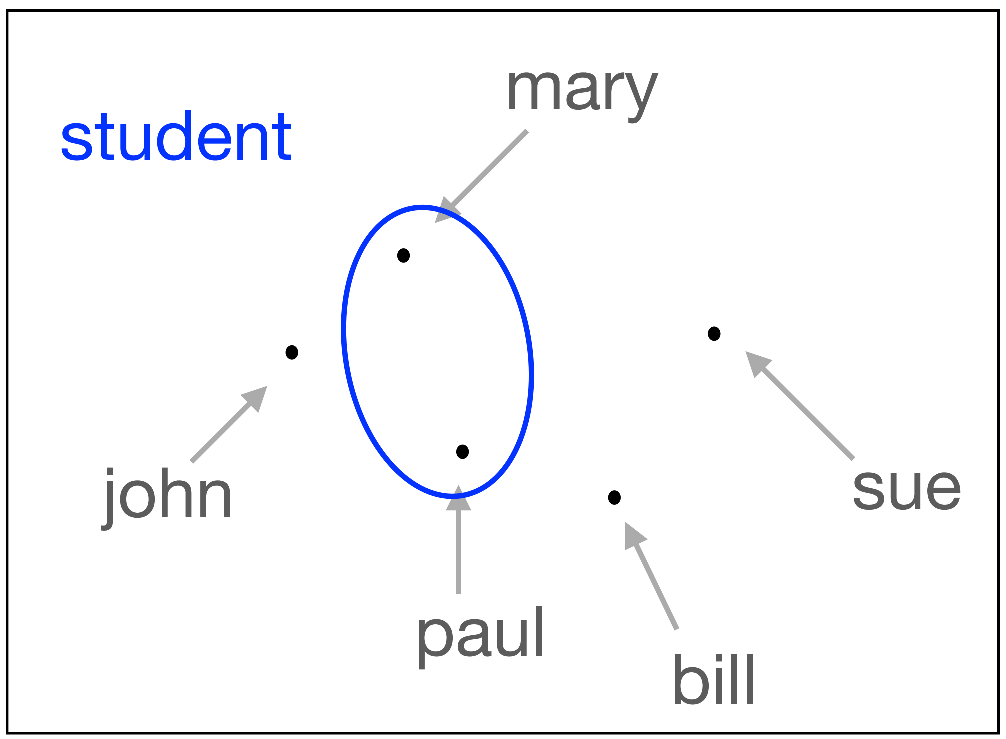
Step 2: Interpretation

Logical models are simplified representations of the state of affairs in the world

M1:



M2:



John is a student : for any M, $[\![\text{student'(john)}]\!]^{M} = 1$ *iff* $V_{M}(\text{john}) \in V_{M}(\text{student'})$

$V_{M1}(\text{john}) \in V_{M1}(\text{student'})$ therefore: $[\![\text{student'(john)}]\!]^{M1} = 1$

$V_{M2}(\text{john}) \notin V_{M2}(\text{student'})$ therefore: $[\![\text{student'(john)}]\!]^{M2} = 0$

A formal description of a model

Model M = ⟨U_M, V_M⟩, with:

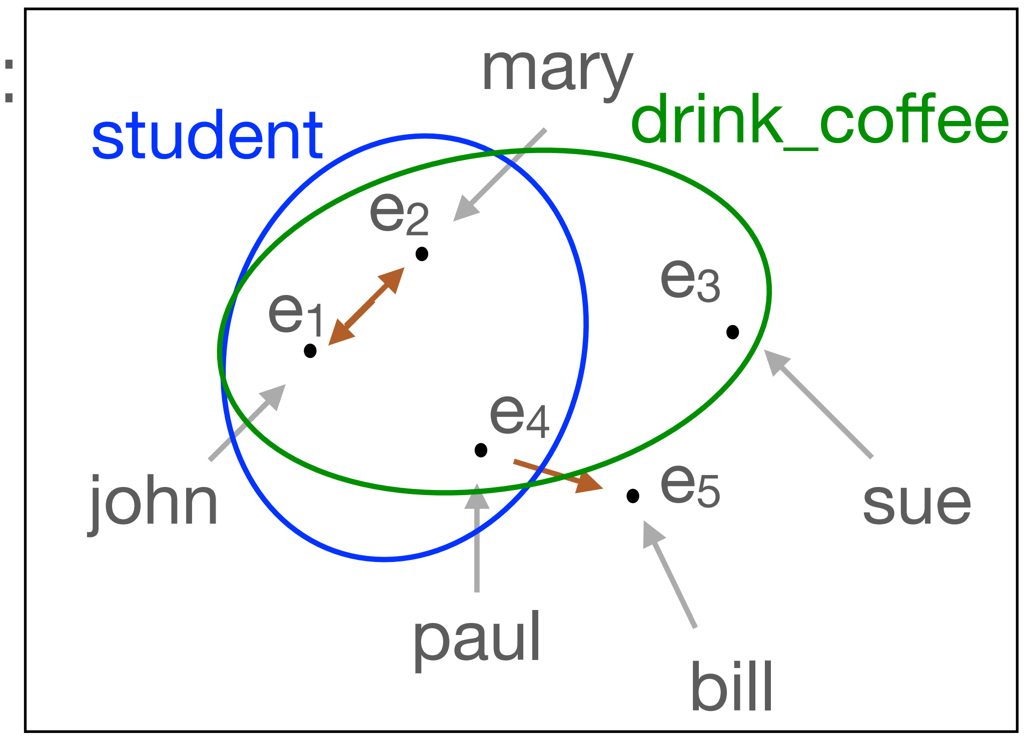
- U_M is the universe of M and
- V_M is an interpretation function

U_M = {e1, e2, e3, e4, e5} **universe**

V_M(john) = e1
… **constants**
V_M(bill) = e5

V_M(student) = {e1, e2, e4}
V_M(drink_coffee) = {e1, e2, e3, e4} **1-place predicates**

V_M(love) = {⟨e1,e2⟩, ⟨e2,e1⟩, ⟨e4,e5⟩} **2-place predicates**

M:


Interpretation in the model

V_M is an interpretation function assigning individuals ($\in U_M$) to individual constants and n-ary relations over U_M to n-place predicate symbols:

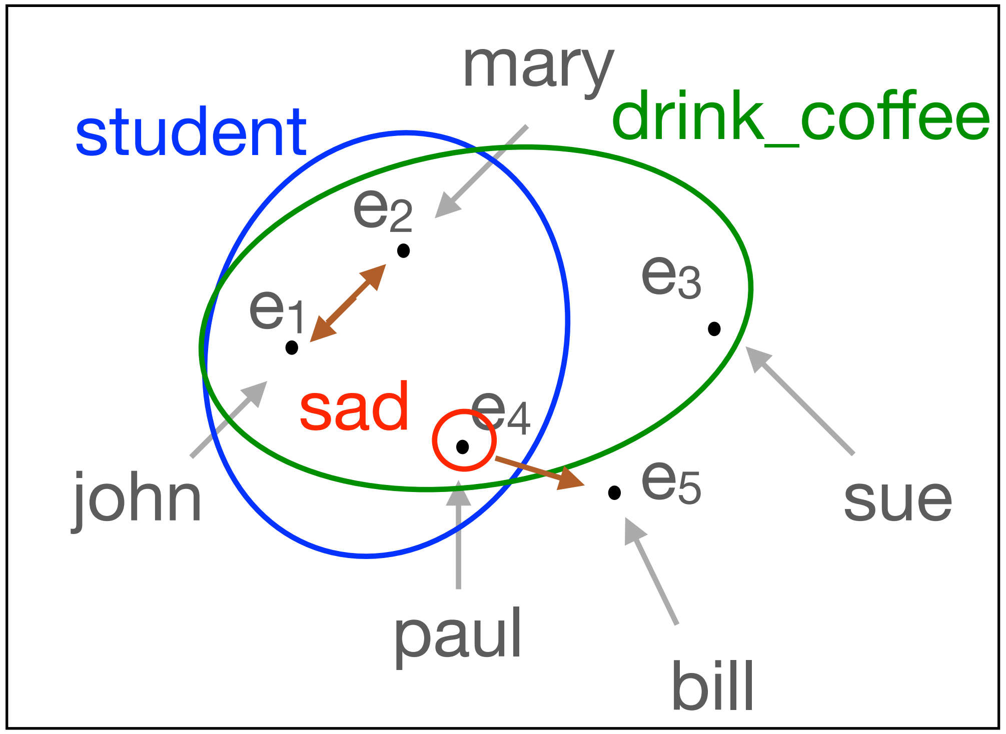
- $V_M(c) \in U_M$ if c is an individual constant
- $V_M(P) \subseteq U_M^n$ if P is an n-place predicate symbol
- $V_M(P) \in \{0,1\}$ if P is an 0-place predicate symbol

Variables and quantifiers

How to interpret the following sentence in our model M:

- Someone is sad ↦ $\exists x(sad'(x))$

M:



Intuition:

- find an entity in the universe for which the statement holds: $V_M(sad') = e_4$

- replace x by e_4 in order to make $\exists x(sad'(x))$ true

More formally:

- Interpret sentence relative to *assignment function* g: i.e., $[\![\exists x(sad'(x))]\!]^{M,g}$, such that $g(x) = e_4$; this can be generalised to any g' as follows: $g'[x/e_4](x) = e_4$

Assignment functions

An assignment function g assigns values to all variables

- g :: VAR → U_M
- We write *g[x/d]* for the assignment function *g'* that assigns *d* to *x* and assigns the same values as *g* to all other variables.

	x	y	z	u	…
g	e_1	e_2	e_3	e_4	…
$g[y/e_1]$	e_1	e_1	e_3	e_4	…
$g[x/e_1]$	e_1	e_2	e_3	e_4	…
$g[y/g(z)]$	e_1	e_3	e_3	e_4	…
$g[y/e_1][u/e_1]$	e_1	e_1	e_3	e_1	…
$g[y/e_1][y/e_2]$	e_1	e_2	e_3	e_4	…

Interpretation of terms

Interpretation of terms with respect to a model *M* and a variable assignment *g*:

$$[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$$

Interpretation of formulas

Interpretation of formulas with respect to a model M and variable assignment g:

- $[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$
- $[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$
- $[\![\neg\phi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$
- $[\![\phi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \vee \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$
- $[\![\phi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$
- $[\![\exists x\phi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\phi]\!]^{M,g[x/d]} = 1$
- $[\![\forall x\phi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\phi]\!]^{M,g[x/d]} = 1$

Truth, Validity and Entailment

A formula ϕ is true in a model M iff:
$[\![\phi]\!]^{M,g} = 1$ for every variable assignment g

A formula ϕ is valid ($\vDash \phi$) iff:
ϕ is true in all models

A formula ϕ is satisfiable iff:
there is at least one model M such that ϕ is true in model M

A set of formulas Γ is (simultaneously) satisfiable iff:
there is a model M such that every formula in Γ is true in M
("M satisfies Γ," or "M is a model of Γ")

Γ entails a formula ϕ ($\Gamma \vDash \phi$) iff:
ϕ is true in every model structure that satisfies Γ

Logical Equivalence

Formula ϕ is logically equivalent to formula ψ ($\phi \Leftrightarrow \psi$), iff:

- $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g}$ for all models M and variable assignments g.

For all *closed* formulas ϕ and ψ, the following assertions are equivalent:

1. $\phi \Leftrightarrow \psi$ (logical equivalence)
2. $\phi \models \psi$ and $\psi \models \phi$ (mutual entailment)
3. $\models \phi \leftrightarrow \psi$ (validity of “material equivalence”)

Logical Equivalence Theorems: Propositions

1)	$\neg\neg\phi \Leftrightarrow \phi$	Double negation
2)	$\phi\wedge\psi \Leftrightarrow \psi\wedge\phi$	Commutativity of $\wedge$, $\vee$
3)	$\phi\vee\psi \Leftrightarrow \psi\vee\phi$	
4)	$\phi\wedge(\psi\vee\chi) \Leftrightarrow (\phi\wedge\psi)\vee(\phi\wedge\chi)$	Distributivity of $\wedge$ and $\vee$
5)	$\phi\vee(\psi\wedge\chi) \Leftrightarrow (\phi\vee\psi)\wedge(\phi\vee\chi)$	
6)	$\neg(\phi\wedge\psi) \Leftrightarrow \neg\phi\vee\neg\psi$	de Morgan's Laws
7)	$\neg(\phi\vee\psi) \Leftrightarrow \neg\phi\wedge\neg\psi$	
8)	$\phi \rightarrow\neg\psi \Leftrightarrow \psi \rightarrow\neg\phi$	Law of Contraposition
9)	$\phi\rightarrow\psi \Leftrightarrow \neg\phi\vee\psi$	
10)	$\neg(\phi \rightarrow \psi) \Leftrightarrow \phi\wedge\neg\psi$	

Logical Equivalence Theorems: Quantifiers

11)	$\neg\forall x\phi \Leftrightarrow \exists x\neg\phi$	Quantifier negation
12)	$\neg\exists x\phi \Leftrightarrow \forall x\neg\phi$	
13)	$\forall x(\phi \wedge \Psi) \Leftrightarrow \forall x\phi \wedge \forall x\Psi$	Quantifier distribution
14)	$\exists x(\phi \vee \Psi) \Leftrightarrow \exists x\phi \vee \exists x\Psi$	
15)	$\forall x\forall y\phi \Leftrightarrow \forall y\forall x\phi$	Quantifier Swap
16)	$\exists x\exists y\phi \Leftrightarrow \exists y\exists x\phi$	
17)	$\exists x\forall y\phi \Rightarrow \forall y\exists x\phi$	... but not vice versa !

Logical Equivalence Theorems: Quantifiers (cont.)

The following equivalences are valid theorems of FOL, provided that x does not occur free in ϕ:

Here, $\phi[x/y]$ is the result of replacing all free occurrences of y in ϕ with x

18) $\exists y \phi \Leftrightarrow \exists x \phi[x/y]$

19) $\forall y \phi \Leftrightarrow \exists x \phi[x/y]$

20) $\phi \wedge \forall x \Psi \Leftrightarrow \forall x(\phi \wedge \Psi)$

21) $\phi \wedge \exists x \Psi \Leftrightarrow \exists x(\phi \wedge \Psi)$

22) $\phi \vee \forall x \Psi \Leftrightarrow \forall x(\phi \vee \Psi)$

23) $\phi \vee \exists x \Psi \Leftrightarrow \exists x(\phi \vee \Psi)$

24) $\phi \rightarrow \forall x \Psi \Leftrightarrow \forall x(\phi \rightarrow \Psi)$

25) $\phi \rightarrow \exists x \Psi \Leftrightarrow \exists x(\phi \rightarrow \Psi)$

26) $\exists x \Psi \rightarrow \phi \Leftrightarrow \forall x(\Psi \rightarrow \phi)$

27) $\forall x \Psi \rightarrow \phi \Leftrightarrow \exists x(\Psi \rightarrow \phi)$

Equivalence Transformations

(1) $\neg\exists x \forall y(Py \rightarrow Rxy)$ "Nobody masters every problem"

(2) $\forall x \exists y(Py \wedge \neg Rxy)$ "Everybody fails to master some problem"

We show the equivalence of (1) and (2) as follows:

$$
\begin{aligned}
\neg\exists x \forall y(Py \rightarrow Rxy) &\Leftrightarrow \forall x \neg\forall y(Py \rightarrow Rxy) && (\neg\exists x\phi \Leftrightarrow \forall x\neg\phi\ ) \\
&\Leftrightarrow \forall x \exists y\neg(Py \rightarrow Rxy) && (\neg\forall x\phi \Leftrightarrow \exists x\neg\phi\ ) \\
&\Leftrightarrow \forall x \exists y(Py \wedge \neg Rxy) && (\neg(\phi \rightarrow \psi) \Leftrightarrow \phi\wedge\neg\psi\ )
\end{aligned}
$$

Background reading material

- Gamut: Logic, Language, and Meaning Vol I/II — Chapter 2
- For a more basic introduction, see:
  http://www.logicinaction.org — Chapter 4