Semantic Theory
Intensional Logic

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Today …

• The principle of extensionality
• Intensional propositional logic
• Intensional predicate logic
Substitutions and Extensionality

• **Substitutions:** If $\phi$ is a subformula of $\chi$, we write $[\psi/\phi]\chi$ for the formula obtained from $\chi$ by replacing $\phi$ in $\chi$ by $\psi$.

• **The principle of extensionality:** If $\phi$ and $\psi$ are equivalent sentences, then $\chi$ and $[\psi/\phi]\chi$ are equivalent.
  - $\Phi \leftrightarrow \Psi \vdash \chi \leftrightarrow [\Phi/\Psi]\chi$
Substitutability?

(1) Barack Obama is married to Michelle Obama.
(2) Barack Obama is the American president.
(3) The American president is married to Michelle Obama.
Substitutability?

(1) Barack Obama has always been married to Michelle Obama.

(2) Barack Obama is the American president.

(3) The American president has always been married to Michelle Obama.
Substitutability?

(1) *In 1963 the president of the United States was assassinated in Dallas, Texas.*

(2) *Barack Obama is the president of the United States.*

(3) *In 1963 Barack Obama was assassinated in Dallas, Texas.*
Substitutability?

(1) By constitution, the American president is the Supreme Commander of the Armed Forces.

(2) Barack Obama is the American president.

(3) By constitution, Barack Obama is the Supreme Commander of the Armed Forces.
Substitutability?

(1) The detective knows that the thief entered through the skylight.

(2) Biggles is the thief.

(3) The detective knows that Biggles entered through the skylight.

say, discover, believe, know, suspect, ...
Substitutability?

(1) *Nine necessarily exceeds seven.*

(2) *Nine is the number of planets.*

(3) *The number of planets necessarily exceeds seven.*
Intensional Logic

• Intensional logic is an extension of “standard” logic (propositional logic and predicate logic)

• Come with “richer” model structures
  ▪ Modal logic – possible worlds
  ▪ Temporal logic – points in time
  ▪ [...]
Modal Propositional Logic

- Extend propositional logic with two modal operators:
  - □A – *it is necessary that* A, *necessarily* A
  - ◇A – *it is possible that* A, *possibly* A
The set FORM of well-formed formulas of propositional modal logic is the smallest set such that:

- All propositional variables are in FORM
- If A, B are in FORM, so are \( \neg A \), (A \( \land \) B), (A \( \lor \) B), (A \( \rightarrow \) B), (A \( \leftrightarrow \) B), \( \Box A \), \( \Diamond A \)
Model Structures

- Model structures: $M = (W, R, V)$
  - $W$ is a non-empty set (of “possible worlds”)
  - $R \subseteq W \times W$ is an accessibility relation on $W$
  - $V$ is value assignment function, which assigns each propositional constant a function $W \to \{0,1\}$
- For $V(p)(w)$ we also write $V_w(p)$ or $V_{M,w}(p)$. 
Model Structures – Example

- M = (U, R, V)
  - U = \{w_1, w_2\}
  - R = \{(w_1, w_1), (w_1, w_2), (w_2, w_1)\}
  - V(p) = \{(w_1, 1), (w_2, 0)\}
Interpretation

\[\[ p \]_{M,w} = 1 \quad \text{iff} \quad V_{M,w}(p) = 1\]
\[\[ \neg \phi \]_{M,w} = 1 \quad \text{iff} \quad [\[ \phi \]_{M,w} = 0\]
\[\[ \phi \land \psi \]_{M,w} = 1 \quad \text{iff} \quad [\[ \phi \]_{M,w} = 1 \text{ and } [\[ \psi \]_{M,w} = 1\]
\[\[ \phi \lor \psi \]_{M,w} = 1 \quad \text{iff} \quad [\[ \phi \]_{M,w} = 1 \text{ or } [\[ \psi \]_{M,w} = 1\]
\[\[ \phi \rightarrow \psi \]_{M,w} = 1 \quad \text{iff} \quad [\[ \phi \]_{M,w} = 0 \text{ or } [\[ \psi \]_{M,w} = 1\]
\[\[ \phi \leftrightarrow \psi \]_{M,w} = 1 \quad \text{iff} \quad [\[ \phi \]_{M,w} = [\[ \psi \]_{M,w}\]
\[\[ \Diamond \phi \]_{M,w} = 1 \quad \text{iff} \quad \exists w' \in W \text{ such that } R(w, w') \text{ and } [\[ \phi \]_{M,w'} = 1\]
\[\[ \Box \phi \]_{M,w} = 1 \quad \text{iff} \quad \forall w' \in W \text{ such that } R(w, w'), [\[ \phi \]_{M,w'} = 1\]
An Example

- \( M = (U, R, V) \)
  - \( U = \{w_1, w_2\} \)
  - \( R = \{(w_1, w_1), (w_1, w_2), (w_2, w_1)\} \)
  - \( V(p) = \{(w_1, 1), (w_2, 0)\} \)

- Which of the following formulae are true in \( w_1, w_2, \) or the whole model?
  1. \( \Box p \to \Box \Box p \)
  2. \( \neg \Box p \)
  3. \( p \to \Box \Diamond p \)
Some Logical Laws

- The following formulae are true for every model structure and possible world.
  1. \( \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \)
  2. \( \Diamond A \leftrightarrow \neg \Box \neg A \)
  3. \( \Box A \leftrightarrow \neg \Diamond \neg A \)
Some Logical Laws

- The following formulae are not true in all model structures, but true in model structures that have a certain structure.

  1. $\Box A \to A$
  2. $A \to \Diamond A$
  3. $\Box A \to \Box \Box A$
  4. $\Diamond \Diamond A \to \Diamond A$
  5. $A \to \Box \Diamond A$
  6. $\Diamond A \to \Box \Diamond A$

- (1) and (2) are true if $R$ is reflexive ("System T")
- (3) and (4) are true if $R$ is transitive ("S4")
- (5) and (6) are true if $R$ is also symmetric ("S5")
Propositional Tense Logic

• Tense logic is a variant of modal logic that deals with time rather than necessity.
  ▪ Usually depends on an accessibility relation that is linear.

• Analogs of □ and ◇:
  ▪ $G\varphi$ – is always going to be the case
  ▪ $H\varphi$ – it has always been the case
  ▪ $F\varphi$ – it will be at some stage in the future be the case that $\varphi$
  ▪ $P\varphi$ – it was be at some stage in the past the case that $\varphi$
Intensional Predicate Logic

• Straightforward syntax:
  ▪ first-order formulae, plus □ and ◇

• Extend model structures with possible worlds

• Interpretation of terms – various options
  ▪ Should every possible world have its own domain?
    ▪ ... or should we assume one fixed domain for every world?
  ▪ Should we interpret constants as individuals: V(c) ∈ U ?
    ▪ ... or as individual concepts: V(c): W → U ?

• (see Gamut 1991, Vol 2, Chapter 3)
Model Structures

• Model structures: $M = (U, W, R, V)$
  - $U$ is a non-empty set (the “universe”)
  - $W$ is a non-empty set (of “possible worlds”)
    - $U \cap W = \emptyset$
  - $R$ is a binary relation on $W$ (the “accessibility relation”)
  - $V$ is value assignment function for non-logical constants
    - $V(a): W \rightarrow U^M$ for individual constants
    - $V(R): W \rightarrow U^n$ for n-place predicate symbols

• Assignment function for variables: $g: \text{VAR} \rightarrow U^M$
Interpretation of Terms

- \( [c]^M,w,g = V_M(w)(c) \) if \( c \) is a constant
- \( [x]^M,w,g = g(x) \) if \( x \) is a variable

- Note: here we interpret constants as “individual concepts”
  - a constant can refer to different individuals in different worlds
Interpretation of Formulae

\[ \llbracket R(t_1, \ldots, t_n) \rrbracket_{M, w, g} = 1 \iff (\llbracket t_1 \rrbracket_{M, w, g}, \ldots, \llbracket t_n \rrbracket_{M, w, g}) \in V_M(w)(R) \]

\[ \llbracket s = t \rrbracket_{M, w, g} = 1 \iff \llbracket s \rrbracket_{M, w, g} = \llbracket t \rrbracket_{M, w, g} \]

\[ \llbracket \neg \varphi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g} = 0 \]

\[ \llbracket \varphi \land \psi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g} = 1 \text{ and } \llbracket \psi \rrbracket_{M, w, g} = 1 \]

\[ \llbracket \varphi \lor \psi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g} = 1 \text{ or } \llbracket \psi \rrbracket_{M, w, g} = 1 \]

\[ \llbracket \varphi \rightarrow \psi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g} = 0 \text{ or } \llbracket \psi \rrbracket_{M, w, g} = 1 \]

\[ \llbracket \exists x \varphi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g[a/x]} = 1 \text{ for some } a \in U_M \]

\[ \llbracket \forall x \varphi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w, g[a/x]} = 1 \text{ for every } a \in U_M \]

\[ \llbracket \square \varphi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w', g} = 1 \text{ for every } w' \text{ with } R(w, w') \]

\[ \llbracket \Diamond \varphi \rrbracket_{M, w, g} = 1 \iff \llbracket \varphi \rrbracket_{M, w', g} = 1 \text{ for at least one } w' \text{ with } R(w, w') \]
Substitutability, revisited

(1) Necessarily, the American president is the Supreme Commander of the Armed Forces.
   ▪ □(∃x(∀y(P(y) ↔ x = y) ∧ C(x)))

(2) Barack Obama is the American president.
   ▪ ∃x(∀y(P(y) ↔ x = y) ∧ x = o)

(3) Necessarily, Barack Obama is the Supreme Commander of the Armed Forces.
   ▪ □C(o)

\[
\begin{align*}
P(x) & \rightarrow x \text{ is the American president} \\
C(x) & \rightarrow x \text{ is the Supreme Commander of the Armed Forces} \\
o & \rightarrow Barack Obama
\end{align*}
\]
Substitutability, revisited

- $\lbrack \exists x (\forall y (P(y) \leftrightarrow x = y) \land C(x) ) \rbrack_{M,w,g} = 1$
  - iff exists $a \in U_M$: $a \in V_M(P)$ and $|V_M(P)| = 1$ and $a \in V_M(C)$

- $\lbrack \Box \exists x (\forall y (P(y) \leftrightarrow x = y) \land C(x) ) \rbrack_{M,w,g} = 1$
  - iff for all $w'$ such that $R(w,w'), \ldots$

- $\lbrack \Box C(o) \rbrack_{M,w,g} = 1$
  - iff for all $w'$ such that $R(w,w'), V_M(o) \in V_M(C)$

|    | W1 | W2 | W3 | W4 | ...
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$R = W \times W$
Literature