Interpretation (terms)

- **Interpretation of terms** with respect to a model structure M and a variable assignment g:
  - $[\![\alpha]\!]^{M,g} = \begin{cases} V_M(\alpha) & \text{if } \alpha \text{ is an individual constant} \\ g(\alpha) & \text{if } \alpha \text{ is a variable} \end{cases}$

Interpretation (atomic formulas)

- **Interpretation of (atomic) formulas** with respect to a model structure M and variable assignment g:

$$[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1 \text{ iff } \langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$$

$$[\![t_1 = t_2]\!]^{M,g} = 1 \text{ iff } [\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$$

Is Vincent a rabbit?

- $[\![\text{rabbit(vincent)}]\!]^{M,g} = 1$
  - iff $[\![\text{vincent}]\!]^{M,g} \in V_M(\text{rabbit})$
  - iff $V_M(\text{vincent}) \in V_M(\text{rabbit})$

$M = (U_M, V_M)$
$U_M = \{ r_1, r_2, h_1, h_2 \}$
$V_M(\text{vincent}) = r_1$
$V_M(\text{mia}) = r_2$
$V_M(\text{rabbit}) = \{ r_1, r_2 \}$
$V_M(\text{white}) = \{ r_2 \}$
$V_M(\text{hat}) = \{ h_1, h_2 \}$
$V_M(\text{in}) = \{ (r_1, h_1) \}$

Interpretation (connectives)

- **Connectives are truth-functional:** the truth-value of a complex expession is determined by the truth-values of their subformulas.

$[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \wedge \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \vee \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

Truth-functional connectives

- A connective is **truth-functional** iff the truth value of any compound statement obtained by applying that connective is a function of the individual truth values of the constituent statements that form the compound.
- **Truth-functional connectives:**
  substituting sub-expressions with the same truth-value does not change the truth of the complete expression.

Truth-functional connectives

(1) *John bumped his head **and** he [John] is crying*

(2) *John bumped his head **and** it is raining*

(3) *John is crying*

(4) *It is raining*

- Assume that (3) and (4) have the same truth-value.
  - Then (1) and (2) must have the same truth-value
  - ***and*** is a truth-functional connective

Truth-functional connectives

(1) *John is crying **because** he [John] bumped his head*

(2) *John is crying **because** it is raining*

(3) *John bumped his head*

(4) *It is raining*

- Assume that (3) and (4) have the same truth-value.
  - (1) and (2) can have different same truth-values⇒
  - ⇒ ***because*** is not truth-functional

Is Vincent a white rabbit?

- $[\![\text{rabbit(vincent)} \wedge \text{white(vincent)}]\!]^{M,g} = 1$
  - iff $[\![\text{rabbit(vincent)}]\!]^{M,g} = 1$ and $[\![\text{white(vincent)}]\!]^{M,g} = 1$
  - iff $V_M(\text{vincent}) \in V_M(\text{rabbit})$ and $V_M(\text{vincent}) \in V_M(\text{white})$

$$
\begin{aligned}
M &= (U_M, V_M) \\
U_M &= \{ r_1, r_2, h_1, h_2 \} \\
V_M(\text{vincent}) &= r_1 \\
V_M(\text{mia}) &= r_2 \\
V_M(\text{rabbit}) &= \{ r_1, r_2 \} \\
V_M(\text{white}) &= \{ r_2 \} \\
V_M(\text{hat}) &= \{ h_1, h_2 \} \\
V_M(\text{in}) &= \{ (r_1, h_1) \}
\end{aligned}
$$

Interpretation (quantifiers)

- **We want:**
  - $[\![\forall x A(x)]\!]^{M,g} = 1$ iff for every $d \in U_M$, $d \in [\![A]\!]^{M,g}$
  - $[\![\exists x A(x)]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $d \in [\![A]\!]^{M,g}$

Interpretation (quantifiers)

- **Interpretation of formulas** with respect to a model structure M and variable assignment g:
  - $[\![\exists x \varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$
  - $[\![\forall x \varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$
- **g[x/d]** is the variable assignment which is identical to g except that it assigns the individual d to variable x.
  - $g[x/d](y) = \begin{cases} d & \text{if } x = y \\ g[x/d](y) = g(y) & \text{if } x \neq y \end{cases}$

Variable assignments

$$g[x/d](y) = \begin{cases} d & \text{if } x = y \\ g[x/d](y) = g(y) & \text{if } x \neq y \end{cases}$$

	x	y	z	u	…
g	a	b	c	d	…
g[x/a]	a	b	c	d	…
g[y/a]	a	a	c	d	…
g[y/g(z)]	a	c	c	d	…
g[y/a][u/a]	a	a	c	a	…
g[y/a][y/b]	a	b	c	d	…

Predicate Logic: Semantics

- **Interpretation of formulas** with respect to a model structure M and variable assignment g:

$[\![R(t_1, \ldots, t_n)]\!]^{M,g} = 1$ iff $\langle [\![t_1]\!]^{M,g}, \ldots, [\![t_n]\!]^{M,g} \rangle \in V_M(R)$

$[\![t_1 = t_2]\!]^{M,g} = 1$ iff $[\![t_1]\!]^{M,g} = [\![t_2]\!]^{M,g}$

$[\![\neg\varphi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$

$[\![\varphi \land \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ and $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \lor \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 1$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \rightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = 0$ or $[\![\psi]\!]^{M,g} = 1$

$[\![\varphi \leftrightarrow \psi]\!]^{M,g} = 1$ iff $[\![\varphi]\!]^{M,g} = [\![\psi]\!]^{M,g}$

$[\![\exists x \varphi]\!]^{M,g} = 1$ iff there is a $d \in U_M$ such that $[\![\varphi]\!]^{M,g[x/d]} = 1$

$[\![\forall x \varphi]\!]^{M,g} = 1$ iff for all $d \in U_M$, $[\![\varphi]\!]^{M,g[x/d]} = 1$

Every rabbit is in a hat

- $[\![\forall x(\text{rabbit}(x) \rightarrow \exists y(\text{hat}(y) \wedge \text{in}(x, y))]\!]^{M, g} = 1$
  - iff … [⇒ whiteboard]

M	$= (U_M, V_M)$
U_M	$= \{ r_1, r_2, h_1, h_2 \}$
$V_M(\text{vincent})$	$= r_1$
$V_M(\text{mia})$	$= r_2$
$V_M(\text{rabbit})$	$= \{ r_1, r_2 \}$
$V_M(\text{white})$	$= \{ r_2 \}$
$V_M(\text{hat})$	$= \{ h_1, h_2 \}$
$V_M(\text{in})$	$= \{ (r_1, h_1) \}$

Not every rabbit is white

- $[\![\neg\forall x(\text{rabbit}(x) \rightarrow \text{white}(x))]\!]^{M, g} = 1$
  - iff … [⇒ whiteboard]

M	$= (U_M, V_M)$
U_M	$= \{ r_1, r_2, h_1, h_2 \}$
$V_M(\text{vincent})$	$= r_1$
$V_M(\text{mia})$	$= r_2$
$V_M(\text{rabbit})$	$= \{ r_1, r_2 \}$
$V_M(\text{white})$	$= \{ r_2 \}$
$V_M(\text{hat})$	$= \{ h_1, h_2 \}$
$V_M(\text{in})$	$= \{ (r_1, h_1) \}$

More Examples

- $[\![\exists x(\forall x\, A(x) \wedge B(x))]\!]^{M, g} = 1$ iff …
- $[\![\forall x\, A(x) \wedge B(x)]\!]^{M, g} = 1$ iff …
- $[\![\exists x\, \forall y\, L(x, y)]\!]^{M, g} = 1$ iff …
- $[\![\forall y\, \exists x\, L(x, y)]\!]^{M, g} = 1$ iff …