

# Computational Linguistics

# Latent Spaces and

# Matrix Factorization

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Summer 2012

## Goal

### Goal:

treat document clustering and word clustering on the same footing (same semantic space)

find low dimensional representations

## The word document matrix

## Clustering

### Document clustering

describe each document by a vector containing the frequencies of the words

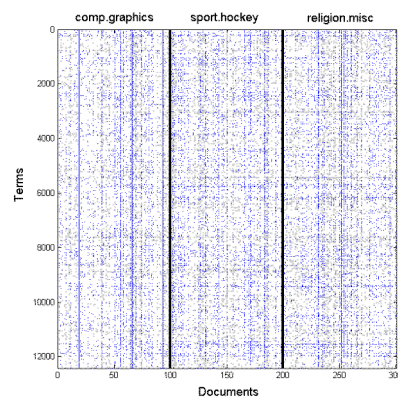
### Word clustering

describe each word by a vector containing the frequencies of its occurrence in different document

## Joint word and document clustering

The word document matrix:

Enter frequency (or tf-idf) for each word and document in a square scheme of numbers (matrix)



**Matrices**

## Matrices

A matrix is an array with two indices

e.g. in a python program this could be `A[i][j]` with  $i=1..N$  and  $j=1..M$

When writing, often a subscript notation is used  $a_{i,j}$

or a square scheme: 
$$A = \begin{pmatrix} a_{1,1} & \dots & a_{1,M} \\ \dots & a_{i,j} & \dots \\ a_{N,1} & \dots & a_{N,M} \end{pmatrix}$$

Specific example of a 2x3 matrix

$$A = \begin{pmatrix} 2 & -5 & 0.5 \\ -2 & 0.1 & -8 \end{pmatrix}$$

## The transpose of a matrix

The two indices are swapped

e.g. in a python program this could be `At[j][i]=A[i][j]` for  $i=1..N$  and  $j=1..M$

for the matrices from the previous slide we have:

$$A^t = \begin{pmatrix} a_{1,1} & \dots & a_{1,N} \\ \dots & a_{j,i} & \dots \\ a_{M,1} & \dots & a_{M,N} \end{pmatrix}$$

Specific example of a 2x3 matrix

$$A = \begin{pmatrix} 2 & -5 & 0.5 \\ -2 & 0.1 & -8 \end{pmatrix}$$

What is

$$A^t$$

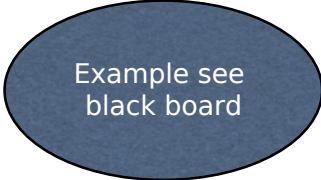
## Product of two matrices

The elements of a product matrix can be calculated in a python program by

```
for i in range(1,N+1):  
    for j in range(1,M+1):  
        for k in range(1,K+1):  
            C[i][j] = A[i][k]*B[k][j]
```

In math notation  $C = A \cdot B$

with 
$$c_{i,j} = \sum_{k=1}^K a_{i,k} b_{k,j}$$



Example see  
black board

## Unit matrix

Unit matrix: the element are the indicator function

$$a_{i,j} = \delta_{i,j}$$

Example:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Often the unit matrix is denoted by a 1

## Orthogonal matrices

a matrix  $A$  is orthogonal if

$$1 = A^t \cdot A$$

Is the following matrix orthogonal:

$$A = \begin{pmatrix} 0.96 & -0.28 \\ 0.28 & 0.96 \end{pmatrix}$$

## Latent Semantic Analysis (LSA)

This section mostly follows Manning and Schütze Chapter 1.

## Singular Value Decomposition

Decompose  $A$  such that

$$\tilde{A} = TSD^t$$

With  $|\tilde{A} - A|^2$  minimal

and

$$T^t \cdot T = 1 \quad D^t \cdot D = 1$$

$A$  a  $t$  by  $d$  matrix     $T$  a  $t$  by  $n$  matrix

$S$  a  $n$  by  $n$  matrix     $D$  a  $d$  by  $n$  matrix

## An artificial Example of Singular Value Decomposition

Is

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$S = (2\sqrt{2})$$

$$D = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

An SVD of

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

## More realistic Example (from Manning and Schütze)

Decompose

$$A = \left( \begin{array}{c|cccccc} & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ \hline \text{cosmonaut} & 1 & 0 & 1 & 0 & 0 & 0 \\ \text{astronaut} & 0 & 1 & 0 & 0 & 0 & 0 \\ \text{moon} & 1 & 1 & 0 & 0 & 0 & 0 \\ \text{car} & 1 & 0 & 0 & 1 & 1 & 0 \end{array} \right)$$

## More realistic Example (from Manning and Schütze)

$$D^t = \left( \begin{array}{c|cccccc} & d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\ \hline \text{Dimension 1} & -0.75 & -0.28 & -0.20 & -0.45 & -0.33 & -0.12 \\ \text{Dimension 2} & -0.29 & -0.53 & -0.19 & 0.63 & 0.22 & 0.41 \\ \text{Dimension 3} & 0.28 & -0.75 & 0.45 & -0.20 & 0.12 & -0.33 \\ \text{Dimension 4} & 0.00 & 0.00 & 0.58 & 0.00 & -0.58 & 0.58 \\ \text{Dimension 5} & -0.53 & 0.29 & 0.63 & 0.19 & 0.41 & -0.22 \end{array} \right)$$

$$T^t = \left( \begin{array}{c|ccccc} & \text{cosm.} & \text{astr.} & \text{moon} & \text{car} & \text{truck} \\ \hline \text{Dimension 1} & -0.44 & -0.13 & -0.48 & -0.70 & -0.26 \\ \text{Dimension 2} & -0.30 & -0.33 & -0.51 & 0.35 & 0.65 \\ \text{Dimension 3} & 0.57 & -0.59 & -0.37 & 0.15 & -0.41 \\ \text{Dimension 4} & 0.58 & 0.00 & 0.00 & -0.58 & 0.58 \\ \text{Dimension 5} & 0.25 & 0.73 & -0.61 & 0.16 & -0.09 \end{array} \right)$$



## More realistic Example (from Manning and Schütze)

$$S = \begin{pmatrix} 2.16 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.59 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.28 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.39 \end{pmatrix}$$

## Document-Document Similarity

Rewrite A

$$A = (d_1 \quad d_2 \quad \dots \quad d_d)$$

with  $d_j$  a vector

with word frequencies of the  $j$ -th document

Similarity of  $i$ -th document with  $j$ -th document  $d_i^t d_j$

All document-document similarities  $d_i^t A$

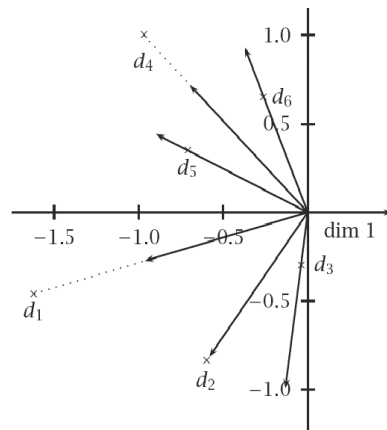
## Document-Document Similarity

$$\begin{aligned}
 \text{Rewrite } \tilde{A}^t \tilde{A} &= \\
 &= (TSD^t)^t TSD^t \\
 &= DS^t T^t TSD^t \\
 &= DS^t SD^t \\
 &= (SD^t)^t SD^t
 \end{aligned}$$

Measure similarity in subspace defined  $SD^t$   
by

## More realistic Example (from Manning and Schütze)

Result for $SD^t$	$d_1$	$d_2$	$d_3$	$d_4$	$d_5$	$d_6$
Dimension 1	-1.62	-0.60	-0.04	-0.97	-0.71	-0.26
Dimension 2	-0.46	-0.84	-0.30	1.00	0.35	0.65



## More realistic Example (from Manning and Schütze)

Decompose  $A$  such that

	$d_1$	$d_2$	$d_3$	$d_4$	$d_5$	$d_6$
$d_1$	1.00					
$d_2$	0.78	1.00				
$d_3$	0.40	0.88	1.00			
$d_4$	0.47	-0.18	-0.62	1.00		
$d_5$	0.74	0.16	-0.32	0.94	1.00	
$d_6$	0.10	-0.54	-0.87	0.93	0.74	1.00

## Term-Term Similarity

$$\begin{aligned}
 \text{Rewrite } \tilde{A}\tilde{A}^t &= \\
 &= (TSD^t)(TSD^t)^t \\
 &= TSD^t DS^t T^t \\
 &= TS^t ST^t \\
 &= (TS)(TS)^t
 \end{aligned}$$

Measure similarity in subspace defined  $TS$   
by

## Task

How does your programming language support SVD

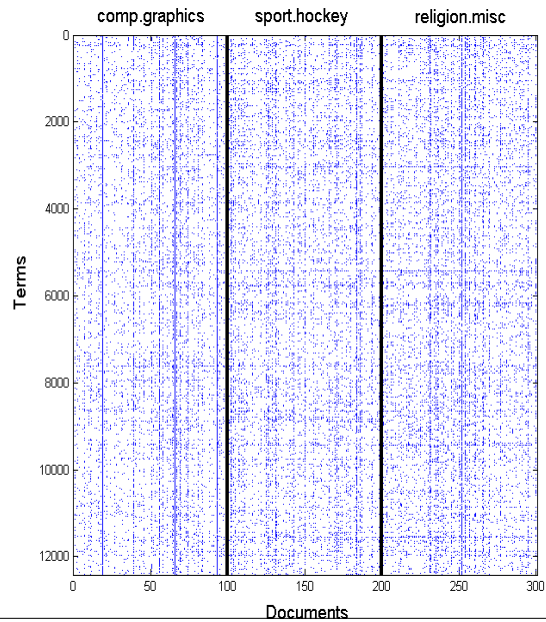
Do some internet search (~10 minutes)

Report your findings

## Homework

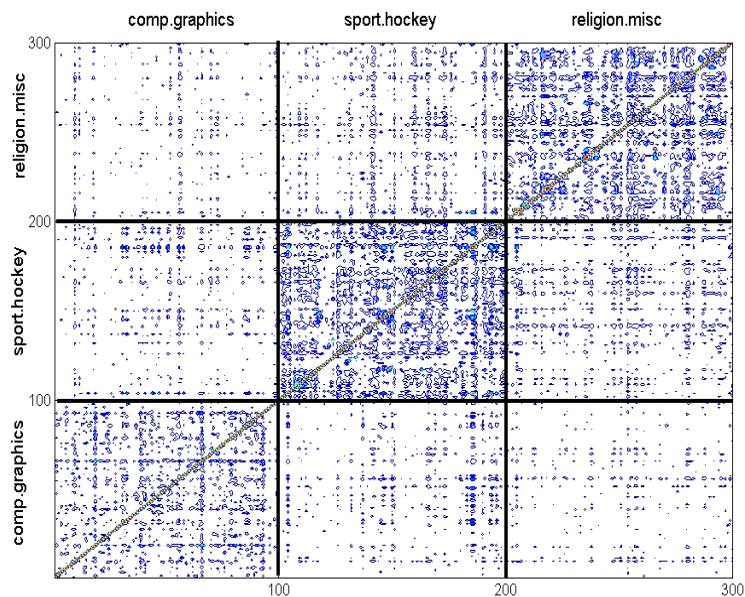
See sheet

## An even more realistic example



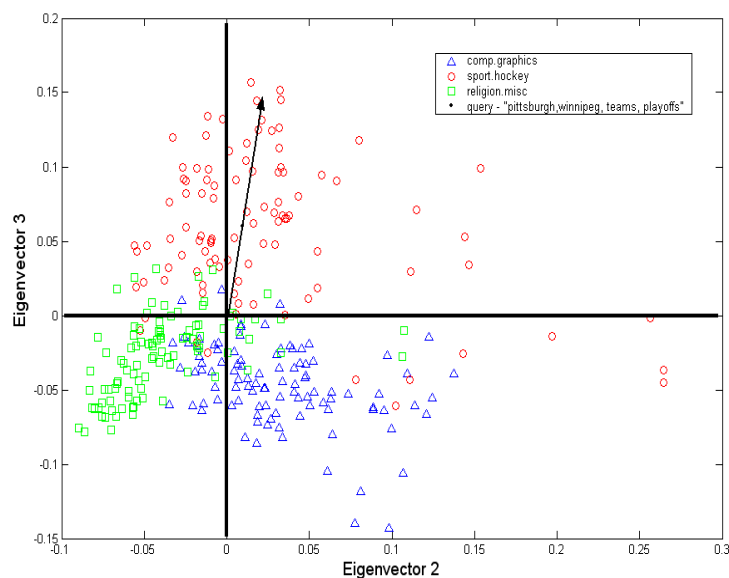
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## An even more realistic example Document-Document Similarity



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## Representation for Documents in 2 dimensional Subspace



## LSA Performance

- LSA consistently improves recall on standard test collections (precision/recall generally improved)
- Variable performance on larger TREC collections
- Dimensionality of Latent Space – a magic number – 300 – 1000 seems to work fine – no satisfactory way of assessing value.
- Computational cost high

## Application (by Landauer et. Al)

How Well Can Passage Meaning be Derived without Using Word Order?  
A Comparison of Latent Semantic Analysis and Humans

Thomas K. Landauer, Darrell Laham, Bob Rehder, and M. E. Schreiner  
Department of Psychology & Institute of Cognitive Science  
University of Colorado, Boulder  
Boulder, CO 80309-0345  
{landauer, dlaham, rehder, mschrey}@psych.colorado.edu

Rate essay by similarity to existing ones  
Measure correlation with human rating

Correlation between	
<u>All Essays (n = 273)</u>	
Two reader scores:	.65
LSA score and average reader score:	.64
<u>Attachment in children (n = 55)</u>	
Two reader scores:	.19
LSA score and average reader score:	.61
<u>Aphasias (n = 109)</u>	
Two reader scores:	.75
LSA score and average reader score:	.60
<u>Operant conditioning (n = 109)</u>	
Two reader scores:	.68
LSA score and average reader score:	.71

Table 2: Psychology essay results.

**Conclusion: drop the right key-words and you are set**

**Probabilistic Latent Semantic Analysis  
(PLSA)**

## Motivation

- Does orthogonality matter?
- Wouldn't a sound statistical foundation be better?

## PLSA

Likelihood of document

$$P(doc) = P(term_1 | doc)P(term_2 | doc) \dots P(term_L | doc)$$

Introduce term-frequency matrix  $X$

$$\prod_{t=1}^L P(term_t | doc) = \prod_{t=1}^T P(term_t | doc)^{A(term_t, doc)}$$



## PLSA

Introduce hidden topic

$$P(\text{term}_t | \text{doc}) = \sum_{k=1}^K P(\text{term}_t | \text{topic}_k) P(\text{topic}_k | \text{doc})$$

Shorthand  $t = \text{term}_t$

$$P(t | \text{doc}) = \sum_{k=1}^K P(t | k) P(k | \text{doc})$$

Relation to LSA?

Likelihood of document

$$P(\text{doc}) = \prod_{t=1}^T \left\{ \sum_{k=1}^K P(t | k) P(k | \text{doc}) \right\}^{A(t, \text{doc})}$$

## PLSA: training

Training objective function

$$\sum_{d=1}^N \log P(d) = \sum_{d=1}^N \sum_{t=1}^T A(t, d) \log \sum_{k=1}^K P(t | k) P(k | d)$$

which is to be maximised w.r.t. parameters  $P(t | k)$  and then also  $P(k | d)$ ,

subject to the constraints that  $\sum_{t=1}^T P(t | k) = 1$  and  $\sum_{k=1}^K P(k | d) = 1$ .

## PLSA: training

### Update term-topic matrix

$$P1(t,k) \leftarrow P1(t,k) \frac{\sum_{d=1}^N A(t,d)}{\sum_{k=1}^K P1(t,k) P2(k,d)} P2(k,d)$$

$$P1(t,k) \leftarrow \frac{P1(t,k)}{\sum_{k=1}^K P1(t,k)}$$

### Update topic-document matrix

$$P2(k,d) \leftarrow P2(k,d) \frac{\sum_{t=1}^T A(t,d)}{\sum_{k=1}^K P1(t,k) P2(k,d)} P1(t,k)$$

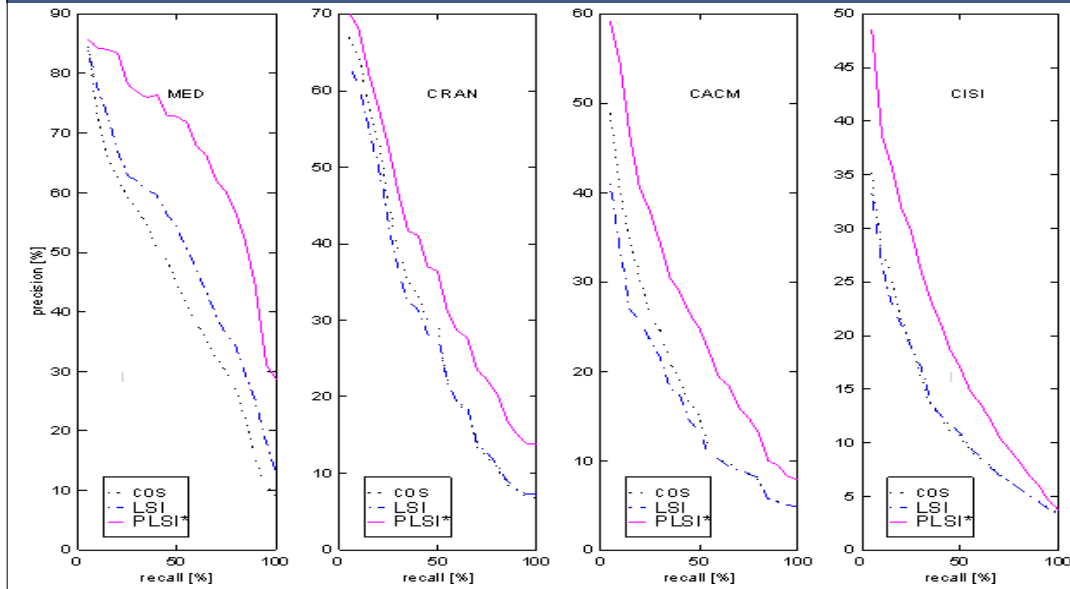
$$P2(k,d) \leftarrow \frac{P2(k,d)}{\sum_{k=1}^K P2(k,d)}$$

## PLSA

### P(t|k) for some

universe	0.0439	drug	0.0672	cells	0.0675	sequence	0.0818	years	0.156
galaxies	0.0375	patients	0.0493	stem	0.0478	sequences	0.0493	million	0.0556
clusters	0.0279	drugs	0.0444	human	0.0421	genome	0.033	ago	0.045
matter	0.0233	clinical	0.0346	cell	0.0309	dna	0.0257	time	0.0317
galaxy	0.0232	treatment	0.028	gene	0.025	sequencing	0.0172	age	0.0243
cluster	0.0214	trials	0.0277	tissue	0.0185	map	0.0123	year	0.024
cosmic	0.0137	therapy	0.0213	cloning	0.0169	genes	0.0122	record	0.0238
dark	0.0131	trial	0.0164	transfer	0.0155	chromosome	0.0119	early	0.0233
light	0.0109	disease	0.0157	blood	0.0113	regions	0.0119	billion	0.0177
density	0.01	medical	0.00997	embryos	0.0111	human	0.0111	history	0.0148
bacteria	0.0983	male	0.0558	theory	0.0811	immune	0.0909	stars	0.0524
bacterial	0.0561	females	0.0541	physics	0.0782	response	0.0375	star	0.0458
resistance	0.0431	female	0.0529	physicists	0.0146	system	0.0358	astrophys	0.0237
coli	0.0381	males	0.0477	einstein	0.0142	responses	0.0322	mass	0.021
strains	0.025	sex	0.0339	university	0.013	antigen	0.0263	disk	0.0173
microbiol	0.0214	reproductive	0.0172	gravity	0.013	antigens	0.0184	black	0.0161
microbial	0.0196	offspring	0.0168	black	0.0127	immunity	0.0176	gas	0.0149
strain	0.0165	sexual	0.0166	theories	0.01	immunology	0.0145	stellar	0.0127
salmonella	0.0163	reproduction	0.0143	aps	0.00987	antibody	0.014	astron	0.0125
resistant	0.0145	eggs	0.0138	matter	0.00954	autoimmune	0.0128	hole	0.00824

## Comparison LSA and PLSA



From Th. Hofmann, 2000

## Non-negative Matrix Factorization

## NMF: the model

- Decomposition of a non-negative matrix  $X$  in two matrices  $W$  and  $H$  both non-negative  
 $A = WH$

- $A$ :  $N \times M$  – data matrix
- $W$ :  $N \times R$  – source matrix
- $H$ :  $R \times M$  – mixture matrix

## NMF: the model

- $W$  und  $H$  werden zufällig mit nicht negativen Werten initialisiert
- Suche  $W$  und  $H$  so, dass das Produkt  $WH$  der Ausgangsmatrix  $A$  möglichst „nahe“ kommt
- Mehrere Strategien möglich:
  - Gradientenabstiegsverfahren
  - Kullback-Leibler-Divergenz  $D(A || WH)$
  - Frobenius-Norm  $\frac{1}{2} \|A - WH\|^2$

## NMF: training

Update

$$H_{ab} = H_{ab} \frac{(W^t A)_{ab}}{(W^t W H)_{ab}}$$
$$W_{ab} = W_{ab} \frac{(A H^t)_{ab}}{(W H H^t)_{ab}}$$

## Summary

Ways to find latent “semantic”  
spaces:

- LSA
- PLSA
- NMF

Similar factorizations

Different target functions and  
constraints